# MA2602 Linear Algebra

Set task - Summer 2013

Attemp all questions

Turn over . . .

# Linear Algebra

In the following questions, $M(2,2)$ denotes the vector space over $\mathbb{R}$ of all real-valued $2 \times 2$ matrices, with $\oplus$ given by matrix addition and $\cdot$ (scalar multiplication) by multiplication of matrices by real numbers. $P_n$ denotes the vector space of all polynomials of degree at most $n$ with real coefficients, with $\oplus$ being addition of polynomials and $\cdot$ multiplication of polynomials by real numbers.

1. (a) Determine whether the following subsets are subspaces (giving reasons for your answers).

   i. $S = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 - x_2 + x_3 - x_4 = 0\}$ in $\mathbb{R}^4$

   ii. $T = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; ab = 0 \right\}$ in $M(2,2)$

   iii. $U = \{a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \in P_n \mid a_0 = a_1\}$ in $P_n$

   (b) Write down two different bases for the real vector space $P_2$.

   (c) Determine which of the following vector sets in $\mathbb{R}^3$ are linearly independent, spanning, or neither. Are any of them a basis for $\mathbb{R}^3$?

   i. $\{(1,2,0), (1,2,3), (6,5,4)\}$

   ii. $\{(1,0,1), (-1,3,-4), (-4,3,5), (3,0,-9)\}$

2. (a) Are the following maps linear? Justify your answers.

   i. $f : \mathbb{R}^3 \to P_2$, with $f(a_0, a_1, a_2) = a_0 + a_1 x + a_2 x^2$.

   ii. $f : \mathbb{R}^2 \to \mathbb{R}^2$, with $f(x, y) = (x - y, y + 1)$

   iii. $f : P_2 \to P_2$, with $f(a_0 + a_1 x + a_2 x^2) = a_2 + a_1 x + a_0 x^2$

   (b) Is there a linear map $f : \mathbb{R}^2 \to \mathbb{R}^3$ such that $f(1,1) = (1,2,-1)$, $f(1,2) = (0,1,0)$ and $f(3,1) = (5,8,-4)$? Justify your answer.

   (c) Define the rank and the nullity of a linear map and state carefully the Rank-Nullity theorem.

   (d) Find bases for the image and the kernel of the linear map $f : P_2 \to P_1$ given by $f(p(x)) = \frac{d}{dx} p(x)$. Based on your results, indicate whether $f$ is injective or surjective.

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3. (a) Let $A$ be a real $n \times n$ matrix. Define what is meant by an eigenvector and an eigenvalue for $A$.

   (b) State the diagonalization theorem for matrices.

   (c) Let $A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$. Use the diagonalization theorem to find an invertible $3 \times 3$ matrix $P$ such that $P^{-1}AP$ is diagonal. Calculate $P^{-1}AP$ and show that its non-zero entries agree with the eigenvalues of $A$.

4. Let $V$ be an inner product space. The *angle* between any two vectors $\mathbf{u}$ and $\mathbf{v}$ in $V$ is defined as

$$\theta = \arccos \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \, \|\mathbf{v}\|}.$$

   (a) Take $V = M(2,2)$ with $\langle A, B \rangle = \mathrm{Tr}(B^T A)$. Calculate the angle between the matrices

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

   (b) Define the notion of an *orthonormal set of vectors*. Do the following vectors in $M(2,2)$ form an orthonormal set?

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, D = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, E = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

   Justify your answer.

   (c) Use the Gram-Schmidt process to construct an orthonormal basis for $M(2,2)$ starting from the basis

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

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