CITY UNIVERSITY
London

BSc Honours Degree in Mathematical Science
Mathematical Science with Statistics
Mathematical Science with Computer Science
Mathematical Science with Finance and Economics
Mathematics and Finance

PART 2

Linear Algebra

2013

Time allowed: 2 hours

Full marks may be obtained for correct answers to FOUR of the FIVE questions.
All necessary working must be shown.

Important note for students regarding past exam papers
Past exam papers are published for illustrative purposes only. They can be used as a study aid but do not provide a definitive guide to either the format of the next exam, the topics that will be examined or the style of questions that will be set. Students should not expect their own exam to be directly comparable with previous papers. Remember that a degree requires an amount of self-study, reading around topics, and lateral thinking particularly at the higher level modules and for higher marks. Specific guidance for your exam will be given by the lecturer.

1. (a) **[10 marks]** Determine whether the following subsets are subspaces giving reasons for your answers.

    i. $\mathcal{U} = \{(x_1, x_2) \mid x_1^2 + x_2^2 = 0\} \subset \mathbb{R}^2$.

    ii. $\mathcal{V} = \{(x_1, x_2) \mid x_1^2 - x_2^2 = 0\} \subset \mathbb{R}^2$.

    iii. $\mathcal{W} = \left\{ A \in M(2,2) \;\middle|\; A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \subset M(2,2)$.

   (b) **[15 marks]** Determine whether the following vector sets are linearly independent, spanning, or neither:

    i. $\{2 + x + x^2,\ 1 + 2x + x^2,\ 1 + x + 2x^2\}$ in P_2;

    ii. $\{(1 - x)^2,\ (1 + x)^2,\ 1 + x^2\}$ in P_2;

    iii. $\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}$ in $M(2,2)$.

2. (a) **[9 marks]** Are the following maps linear? Justify your answers.

    i. $f : \mathbb{R}^2 \mapsto P_2$, with $f(a, b) = (x - a)^2 + (x - b)^2$

    ii. $g : P_2 \to M(2,2)$, with $g(p(x)) = \begin{pmatrix} p(0) & p(1) \\ p(1) & p(2) \end{pmatrix}$

    iii. $h : M(2,2) \to \mathbb{R}^2$, with $h(M) = M \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

   (b) **[8 marks]** Give the general expression $h(x, y, z)$ for the linear map $h : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $h(1, 1, 1) = (2, 2, 0)$, $h(1, 2, 1) = (3, 3, 0)$ and $h(1, 0, 0) = (1, 0, 1)$.

   (c) **[8 marks]** Find bases for the image and the kernel of the linear map h from (b). Then state the Rank-Nullity theorem for linear maps and verify it for h.

3. Consider the linear map

$$g \,:\, P_2 \longrightarrow P_2 \;:\; p(x) \mapsto p(1) + \frac{d}{dx}(p(x))$$

for all $p(x) \in P_2$.

(a) **[5 marks]** Find $g(1), g(x)$ and $g(x^2)$ and hence find the matrix A representing g with respect to the ordered basis $\{1, x, x^2\}$ of P_2.

(b) **[10 marks]** Find $g(1+x)$ and $g(1+x+x^2)$ and hence find the matrix B representing g with respect to the ordered basis $\{1, 1+x, 1+x+x^2\}$ of P_2.

(c) **[10 marks]** Express each vector $1, 1+x$ and $1+x+x^2$ in coordinate form with respect to the ordered basis $\{1, x, x^2\}$ and hence find the change of basis matrix P and check that it satisfies the Change of Basis Theorem, namely $P^{-1}AP = B$.

4. (a) **[10 marks]** Consider the following matrix, in which c is a constant:

$$M = \begin{pmatrix} 1 & 1 \\ c & 1 \end{pmatrix}.$$

State the diagonalisation theorem for matrices and discuss for which values of c is M diagonalisable over $\mathbb{R}$.

(b) **[15 marks]** Calculate the eigenvalues and eigenvector subspaces of the matrix

$$A = \begin{pmatrix} 2 & -3 & 3 \\ 3 & -4 & 3 \\ 6 & -6 & 5 \end{pmatrix}.$$

Is A is diagonalisable? If it is then find an appropriate change of basis matrix P and show by explicit calculation that $P^{-1}AP$ is diagonal. It it is not, explain in detail why.

5. (a) **[10 marks]** For any two vectors $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$ define the map

$$\langle \mathbf{x}, \mathbf{y} \rangle = (x_1 + x_2)(y_1 + y_2) + (x_1 - x_2)(y_1 - y_2).$$

Is this map an inner product in $\mathbb{R}^2$? Justify your answer.

(b) **[15 marks]** Consider the vector set

$$\mathcal{S} = \left\{ 1 + x, \quad 1 + \frac{1}{3}\,x^2, \quad \frac{1}{2}\,x + \frac{1}{3}\,x^2 \right\},$$

which is a basis of P_2. By means of the Gram-Schmidt procedure, construct from $\mathcal{S}$ a basis of P_2 that is orthonormal with respect to the inner product

$$\langle p(x), q(x) \rangle = p_0 q_0 + 2 p_1 q_1 + 3 p_2 q_2$$

in which $p(x) = p_0 + p_1 x + p_2 x^2$ and $q(x) = q_0 + q_1 x + q_2 x^2$.

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