MA2602 Linear Algebra. Solutions to 2013 paper

All questions are **unseen** except for the definitions in 2c and 4a.

1. (a) i. [**2 marks**] Note that $\mathcal{U} = \{(0,0)\}$ which is obviously a subspace of $\mathbb{R}^2$.

   ii. [**4 marks**] Even though $\mathcal{V}$ contains $(0,0)$ and for all $x \in \mathcal{V}$ and $\lambda \in \mathbb{R}$ we have $\lambda\, x \in \mathcal{V}$, it fails to contain the sum of any two of its elements. For example, $(1,1)$ and $(1,-1)$ are in $\mathcal{V}$ but their sum $(2,0)$ is not, so $\mathcal{V}$ is not a subspace of $\mathbb{R}^2$.

   iii. [**4 marks**] The subset $\mathcal{W}$ contains the 2×2 zero matrix, and for every A and B in $\mathcal{W}$ and $\lambda \in \mathbb{R}$ we have $A + B$ and λA in $\mathcal{W}$. Therefore $\mathcal{W}$ is a subspace of $M(2,2)$.

   (b) i. [**5 marks**] Linearly independent because the only solution to $\alpha(2 + x + x^2) + \beta(1 + 2x + x^2) + \gamma(1 + x + 2x^2) = 0$ is $\alpha = \beta = \gamma = 0$. Being a maximal independent set in P_2, it is also spanning and therefore a basis.

   ii. [**5 marks**] Linearly dependent: $(1-x)^2 + (1+x)^2 = 2(1+x^2)$. Not spanning.

   iii. [**5 marks**] Linearly dependent. Note for example that

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

   Also, not spanning.

2. (a) i. [**2 marks**] f is not linear. Note *e.g.* that $f(0,0) = 2x^2$ instead of the zero polynomial.

   [**4 marks**] g is linear. Check the two conditions. For any $a_0 + a_1x + a_2x^2$ and $b_0 + b_1x + b_2x^2 \in P_2$ and $\lambda \in \mathbb{R}$:

   A. $g((a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2))$
   $= g(a_0 + b_0 + (a_1 + b_1)x + (a_2 + b_2)x^2) =$
   $\begin{pmatrix} a_0 + b_0 & a_0 + b_0 + a_1 + b_1 + a_2 + b_2 \\ a_0 + b_0 + a_1 + b_1 + a_2 + b_2 & a_0 + b_0 + 2(a_1 + b_1) + 4(a_2 + b_2) \end{pmatrix}$
   $= g(a_0 + a_1x + a_2x^2) + g(b_0 + b_1x + b_2x^2)$;

   B. $g(\lambda(a_0 + a_1x + a_2x^2)) = g(\lambda a_0 + \lambda a_1x + \lambda a_2x^2)$
   $= \begin{pmatrix} \lambda a_0 & \lambda a_0 + \lambda a_1 + \lambda a_2 \\ \lambda a_0 + \lambda a_1 + \lambda a_2 & \lambda a_0 + 2\lambda a_1 + 4\lambda a_2 \end{pmatrix}$
   $= \lambda g(a_0 + a_1x + a_2x^2)$.

ii. [**3 marks**] h is linear. Check two conditions:

A. $h(M_1 + M_2) = (M_1 + M_2)\begin{pmatrix}1\\0\end{pmatrix} = M_1\begin{pmatrix}1\\0\end{pmatrix} + M_2\begin{pmatrix}1\\0\end{pmatrix}$
$= h(M_1) + h(M_2)$;

B. $h(\lambda M) = (\lambda M)\begin{pmatrix}1\\0\end{pmatrix} = \lambda h(M)$.

(b) [**8 marks**] Decompose (x, y, z) in the basis $(1, 1, 1), (1, 2, 1)$ and $(1, 0, 0)$:

$$(x, y, z) = (-y + 2z)(1, 1, 1) + (y - z)(1, 2, 1) + (x - z)(1, 0, 0).$$

Then linearity implies $h(x, y, z) = (x + y, y + z, x - z)$.

(c) [**8 marks**] For any linear map $f : V \to W$ the rank of f (written $\text{Rank}(f)$) is the dimension of $\text{Img}(f)$ and the nullity of f, written $\text{Null}(f)$, is the dimension of $\text{Ker}(f)$. For the linear map in question, $\text{Ker}\, h = \{(x, -x, x) \,|\, x \in \mathbb{R}\}$ with basis $\{(1, -1, 1)\}$, for example. Then $\text{Null}(h) = 1$. Now $\text{Img}\, h = \{(a + b, b, a) \,|\, a, b \in \mathbb{R}\}$ with basis $\{(1, 0, 1), (1, 1, 0)\}$ for example, so $\text{Rank}(h) = 2$. The Rank-Nullity Theorem states that

$$\dim(V) = \text{Rank}(h) + \text{Null}(h).$$

Because $V = \mathbb{R}^3$ has dimension three the Rank-Nullity theorem is satisfied: 3=2+1.

3. (a) [**5 marks**] We have $g(1) = 1 + 0 = 1 = 1.1 + 0x + 0x^2$, $g(x) = 1 + 1 = 2 = 2.1 + 0x + 0x^2$ and $g(x^2) = 1 + 2x = 1.1 + 2x + 0x^2$. Thus the matrix representing g with respect to the basis $\{1, x, x^2\}$ is given by

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

(b) [**10 marks**] We have $g(1) = 1 = 1.1 + 0(1 + x) + 0(1 + x + x^2)$, $g(1 + x) = 3 = 3.1 + 0(1 + x) + 0(1 + x + x^2)$ and $g(1 + x + x^2) = 4 + 2x = 2.1 + 2(1 + x) + 0(1 + x + x^2)$. Thus the matrix representing g with respect to the basis $\{1, 1 + x, 1 + x + x^2\}$ is given by

$$B = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

(c) **[10 marks]** We have $1 = 1.1 + 0x + 0x^2$, $1 + x = 1.1 + 1x + 0x^2$ and $1 + x + x^2 = 1.1 + 1x + 1x^2$. Thus the change of basis matrix is given by

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

We have

$$P^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

and we check that

$$\begin{aligned} P^{-1}AP &= \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 3 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = B \end{aligned}$$

as required.

4. (a) **[10 marks]** The matrix is diagonalisable over $\mathbb{R}$ if and only if there is a basis of $\mathbb{R}^2$ consisting of eigenvectors of M with real eigenvalues. The characteristic polynomial of M is $(1-\lambda)^2 - c$, so the eigenvalues are $\lambda_\pm = 1 \pm \sqrt{c}$. Therefore the first requirement for M to be diagonalisable over $\mathbb{R}$ is that $c \geq 0$.

The eigenvector subspace corresponding to λ_+ is

$$S_M(\lambda_+) = \{\mathbf{v} \in \mathbb{R}^2 \,|\, (M - \lambda_+)\mathbf{v} = \mathbf{0}\},$$

which for $\mathbf{v} = (v_1, v_2)$ requires $v_2 = \sqrt{c}\, v_1$. Therefore

$$S_M(\lambda_+) = \{(x, \sqrt{c}\, x) \,|\, x \in \mathbb{R}\}$$

generated by $(1, \sqrt{c})$. Similarly

$$S_M(\lambda_-) = \{\mathbf{v} \in \mathbb{R}^2 \,|\, (M - \lambda_-)\mathbf{v} = \mathbf{0}\} = \{(x, -\sqrt{c}\, x) \,|\, x \in \mathbb{R}\}$$

Generated by $(1, -\sqrt{c})$. The matrix M will be diagonalisable if and only if $\{(1, \sqrt{c}), (1, -\sqrt{c})\}$ constitutes a basis, which it will if and only if $c \neq 0$.

In conclusion, M is diagonalisable if and only if $c > 0$.

(b) **[15 marks]** Solving the characteristic equation of the matrix A gives the eigenvalues $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = 5$. The eigenvector subspaces are

$$\begin{aligned} S_A(-1) &= \{(y-z, y, z) \mid y, z \in \mathbb{R}\} \quad \text{with basis } \{(1,1,0), (-1,0,1)\}; \\ S_A(5) &= \{(x, x, 2x) \mid x \in \mathbb{R}\} \quad \text{with basis } \{(1,1,2)\}. \end{aligned}$$

Arranging the coordinates of (a choice of) eigenvector basis into columns gives the change of basis matrix P:

$$P = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \text{ with inverse } P^{-1} = \begin{pmatrix} -1/2 & 3/2 & -1/2 \\ -1 & 1 & 0 \\ 1/2 & -1/2 & 1/2 \end{pmatrix}.$$

Direct calculation shows that $P^{-1}AP$ is diagonal, with diagonal entries given by $-1, -1, 5$.

5. (a) **[10 marks]** Symmetry and linearity in the two variables is obvious. Positivity is also clear:

$$\langle \mathbf{x}, \mathbf{x} \rangle = (x_1 + x_2)^2 + (x_1 - x_2)^2 \geq 0.$$

Also, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ requires $x_1 - x_2 = 0$ and $x_1 + x_2 = 0$. Hence $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only $\mathbf{x} = \mathbf{0}$. In conclusion, it is an inner product.

(b) **[15 marks]** Start by defining $\mathbf{w}_1 = 1 + x$, which is of norm $\sqrt{3}$ with respect to the stated inner product. Then define

$$\mathbf{v}_1 = \frac{1}{\sqrt{3}}(1+x).$$

Now take

$$\begin{aligned} \mathbf{w}_2 &= 1 + \tfrac{1}{3}x^2 - \left\langle 1 + \tfrac{1}{3}x^2, \mathbf{v}_1 \right\rangle \mathbf{v}_1 = 1 + \tfrac{1}{3}x^2 - \frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}}(1+x) \\ &= \frac{1}{3}\left(2 - x + x^2\right), \end{aligned}$$

which is of norm one in the given inner product. Then

$$\mathbf{v}_2 = \mathbf{w}_2 / ||\mathbf{w}_2|| = \frac{1}{3}\left(2 - x + x^2\right).$$

Finally, define

$$\begin{aligned} \mathbf{w}_3 &= \tfrac{1}{2}x + \tfrac{1}{3}x^2 - \left\langle \tfrac{1}{2}x + \tfrac{1}{3}x^2, \mathbf{v}_1 \right\rangle \mathbf{v}_1 - \left\langle \tfrac{1}{2}x + \tfrac{1}{3}x^2, \mathbf{v}_2 \right\rangle \mathbf{v}_2 \\ &= \tfrac{1}{2}x + \tfrac{1}{3}x^2 - \frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}}(1+x) - 0 = \frac{1}{3}(-1 + \tfrac{1}{2}x + x^2), \end{aligned}$$

which has norm $||\mathbf{w}_3|| = 1/\sqrt{2}$. Then

$$\mathbf{v}_3 = \frac{\sqrt{2}}{3}\left(-1 + \frac{1}{2}\,x + x^2\right).$$

In conclusion, the orthonormal basis is

$$\left\{\frac{1}{\sqrt{3}}(1+x), \quad \frac{1}{3}\left(2 - x + x^2\right), \quad \frac{\sqrt{2}}{3}\left(-1 + \frac{1}{2}\,x + x^2\right)\right\}.$$