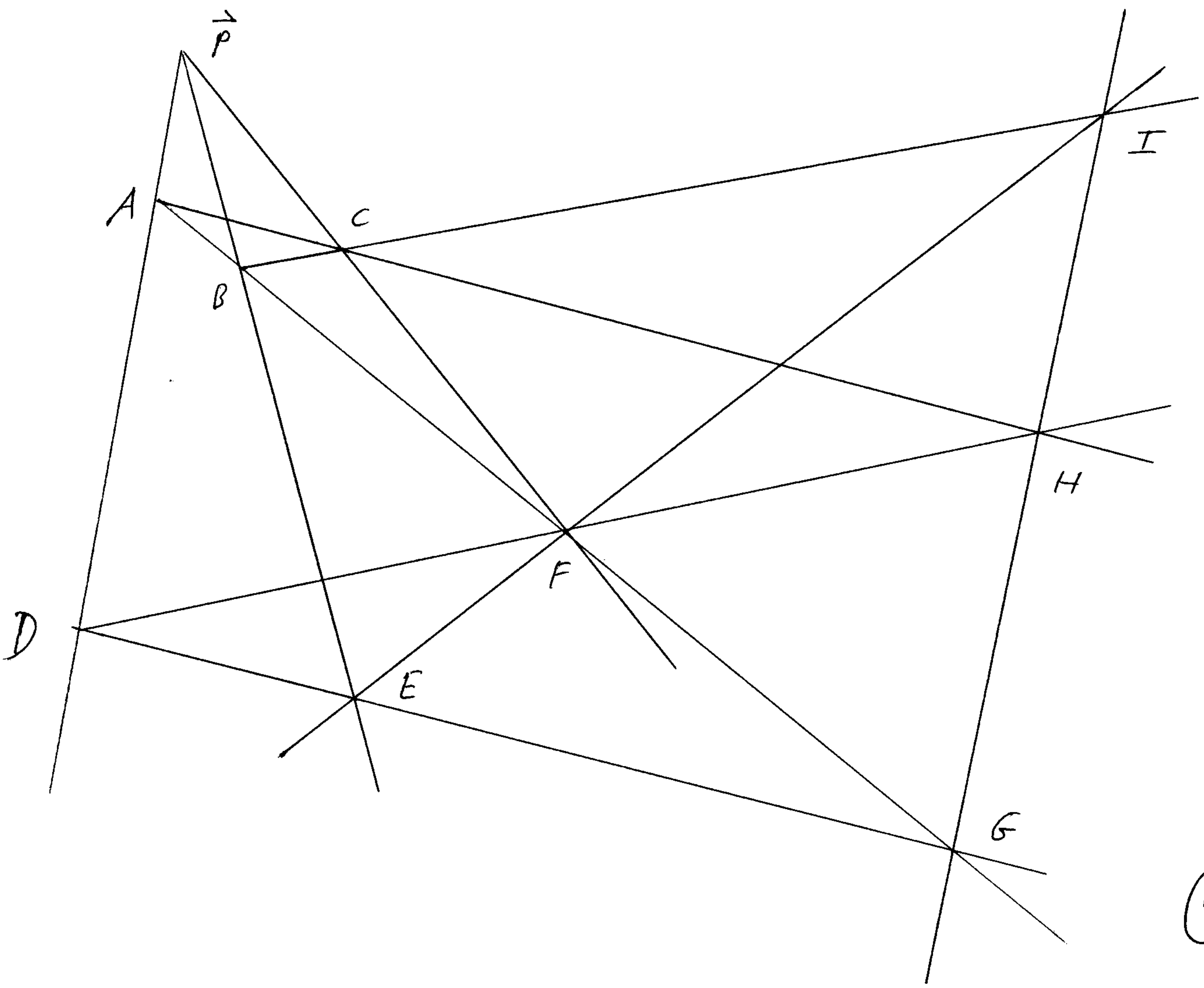
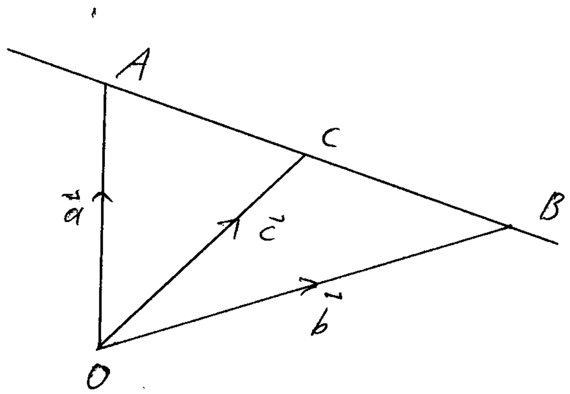
Geometry & Vectors CW1 (2008)

1) i)



④

ii) Consider



$\vec{a} = \overrightarrow{OA}$
$\vec{b} = \overrightarrow{OB}$
$\vec{c} = \overrightarrow{OC}$

$\overrightarrow{AB} = \vec{b} - \vec{a}$ $\quad$ $\overrightarrow{AC} = \vec{c} - \vec{a}$ $\quad$ $\overrightarrow{AB} \parallel \overrightarrow{AC} \Rightarrow \overrightarrow{AC} = \lambda \overrightarrow{AB}$

$\Rightarrow \vec{c} - \vec{a} = \lambda(\vec{b} - \vec{a}) \quad \Rightarrow \vec{c} = \lambda \vec{b} + (1-\lambda)\vec{a}$ ③

From figure in i):

$$\vec{p} = \alpha \vec{a} + (1-\alpha)\vec{d}$$
$$= \beta \vec{b} + (1-\beta)\vec{e} \qquad (*)$$
$$= \gamma \vec{c} + (1-\gamma)\vec{f}$$

$(*) \Rightarrow \alpha \vec{a} - \beta \vec{b} = -(1-\alpha)\vec{d} + (1-\beta)\vec{e}$

$$\Rightarrow \frac{\alpha \vec{a} - \beta \vec{b}}{\alpha - \beta} = -\frac{1-\alpha}{\alpha-\beta}\vec{d} + \frac{1-\beta}{\alpha-\beta}\vec{e}$$

$$\Rightarrow \vec{g} = \frac{\alpha}{\alpha-\beta}\vec{a} + \left(1 - \frac{\alpha}{\alpha-\beta}\right)\vec{b} \Rightarrow G, A, B \text{ are collinear}$$

$$= -\frac{1-\alpha}{\alpha-\beta}\vec{d} + \left(1 - \frac{1-\alpha}{\alpha-\beta}\right)\vec{e} \Rightarrow G, D, E \text{ are collinear}$$

— from equality $\overleftrightarrow{AB}$ and $\overleftrightarrow{DE}$ intersect in G

$$(*) \Rightarrow \frac{\alpha \vec{a} - \gamma \vec{c}}{\alpha-\gamma} = -\frac{(1-\alpha)}{\alpha-\gamma}\vec{d} + \frac{(1-\gamma)}{\alpha-\gamma}\vec{f}$$

$$\Rightarrow h = \frac{\alpha}{\alpha-\gamma}\vec{a} + \left(1 - \frac{\alpha}{\alpha-\gamma}\right)\vec{c} \Rightarrow A, C, H \text{ are collinear}$$

$$= -\frac{(1-\alpha)}{\alpha-\gamma}\vec{d} + \left(1 + \frac{1-\alpha}{\alpha-\gamma}\right)\vec{f} \Rightarrow D, F, H \text{ are collinear}$$

— from equality $\overleftrightarrow{AC}$ and $\overleftrightarrow{DF}$ intersect in H

$$(*) \Rightarrow \frac{\beta \vec{b} - \gamma \vec{c}}{\beta-\gamma} = -\frac{(1-\beta)}{\beta-\gamma}\vec{e} + \frac{(1-\gamma)}{\beta-\gamma}\vec{f}$$

$$\Rightarrow \vec{i} = \frac{\beta}{\beta-\gamma}\vec{b} + \left(1 - \frac{\beta}{\beta-\gamma}\right)\vec{c} \Rightarrow B, C, I \text{ are collinear}$$

$$= -\frac{1-\beta}{\beta-\gamma}\vec{e} + \left(1 + \frac{1-\beta}{\beta-\gamma}\right)\vec{f} \Rightarrow E, F, I \text{ are collinear}$$

- from equality $\overleftrightarrow{BC}$ and $\overleftrightarrow{EF}$ intersect in I (10)

We have
$$\left.\begin{aligned} (\alpha-\beta)\vec{g} &= \alpha\vec{a} - \beta\vec{b} \\ (\alpha-\gamma)\vec{h} &= \alpha\vec{a} - \gamma\vec{c} \\ (\beta-\gamma)\vec{i} &= \beta\vec{b} - \gamma\vec{c} \end{aligned}\right\} \Rightarrow (\alpha-\beta)\vec{g} - (\alpha-\gamma)\vec{h} + (\beta-\gamma)\vec{i} = 0$$

$$\Rightarrow g = \frac{\alpha-\gamma}{\alpha-\beta}\vec{h} \underbrace{- \frac{\beta-\gamma}{\alpha-\beta}}_{1 - \frac{\alpha-\gamma}{\alpha-\beta}}\vec{i} \Rightarrow G, H, I \text{ are collinear}$$ (3)

$\boxed{\Sigma = 20}$

We have $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$

$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ $\qquad \vec{u}\cdot\vec{v} = (-\frac{44}{7})(-2) + \lambda + 11 = \frac{165}{7} + \lambda$

$|\vec{u}| = \sqrt{\vec{u}\cdot\vec{u}} = \sqrt{\frac{2475}{49} + \lambda^2}$ $\qquad |\vec{v}| = 4$

$\Rightarrow \frac{\sqrt{3}}{2} = \dfrac{\lambda + 165/7}{\sqrt{\frac{2475}{49} + \lambda^2}\cdot 4}$ $\qquad \Rightarrow \underline{\underline{\lambda = \frac{15}{7}}}$ (4)

$\vec{w} = a\vec{i} + b\vec{j} + c\vec{k}$

$$\left.\begin{array}{lll} \vec{w}\perp\vec{u}: & \vec{w}\cdot\vec{u} = -\frac{44}{7}a + b + \sqrt{11}\,c & = 0 \\ \vec{w}\perp\vec{v}: & \vec{w}\cdot\vec{v} = -2a + b + \sqrt{11}\,c & = 0 \\ |\vec{w}|: & 1 = a^2 + b^2 + c^2 & \end{array}\right\} \quad a = 0, \quad b = \pm\frac{1}{2}\sqrt{\frac{11}{3}}, \quad c = \mp\frac{1}{2\sqrt{3}}$$

(4)

polarisation identity $\quad \vec{u}\cdot\vec{v} = \frac{1}{2}\left(|\vec{u}|^2 + |\vec{v}|^2 - |\vec{u}-\vec{v}|^2\right)$

$|\vec{u}|^2 = \frac{2475}{49}$ $\qquad |\vec{v}|^2 = 16$

$\vec{u} - \vec{v} = -\frac{30}{7}\vec{i} - \vec{j}$ $\quad \Rightarrow \quad |\vec{u}-\vec{v}|^2 = \frac{949}{49}$

$\Rightarrow \vec{u}\cdot\vec{v} = \frac{165}{7} = \frac{1}{2}\left(\frac{2475}{49} + 16 - \frac{949}{49}\right)$ (3)

$$\frac{7}{\sqrt{11}}\,\vec{u}\times\vec{v} = \frac{7}{\sqrt{11}}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{44}{7} & \frac{22}{7} & \sqrt{11} \\ -2 & 1 & \sqrt{11} \end{vmatrix} = \frac{7}{\sqrt{11}}\begin{vmatrix} \vec{i}+2\vec{j} & \vec{j} & \vec{k} \\ 0 & \frac{22}{7} & \sqrt{11} \\ 0 & 1 & \sqrt{11} \end{vmatrix}$$

$$= \frac{7}{\sqrt{11}}(\vec{i}+2\vec{j})\begin{vmatrix} 22/7 & \sqrt{11} \\ 1 & \sqrt{11} \end{vmatrix}$$

$$= \frac{7}{\sqrt{11}}\left(\frac{22}{7}\sqrt{11} - \sqrt{11}\right)(\vec{i}+2\vec{j})$$

$$= (22-7)(\vec{i}+2\vec{j}) = \underline{\underline{15\vec{i} + 30\vec{j}}}$$ (4)

$\boxed{T = 15}$

$\vec{\ell}_1 = \vec{\ell}_2 \iff (1)\quad \vec{a} + \lambda \vec{u} = \vec{b} + \mu \vec{v} \quad | \times \vec{u}$

$\vec{v} \cdot | \quad \vec{a} \times \vec{u} + \lambda \underbrace{\vec{u} \times \vec{u}}_{0} = \vec{b} \times \vec{u} + \mu \vec{v} \times \vec{u}$

$\vec{v} \cdot (\vec{a} \times \vec{u}) = \vec{v} \cdot (\vec{b} \times \vec{u}) + \mu \underbrace{\vec{v} \cdot (\vec{v} \times \vec{u})}_{0}$ 3

$(1) \times \vec{v}$:

$\vec{b} \cdot | \quad \vec{a} \times \vec{v} + \lambda \vec{u} \times \vec{v} = \vec{b} \times \vec{v} + \mu \underbrace{\vec{v} \times \vec{v}}_{0}$

$\vec{b} \cdot (\vec{a} \times \vec{v}) + \lambda \vec{b} \cdot (\vec{u} \times \vec{v}) = \underbrace{\vec{b} \cdot (\vec{b} \times \vec{v})}_{0}$

$\Rightarrow \lambda = - \dfrac{\vec{b} \cdot (\vec{a} \times \vec{v})}{\vec{b} \cdot (\vec{u} \times \vec{v})} = \dfrac{\vec{a} \cdot \vec{b} \times \vec{v}}{\vec{v} \cdot \vec{b} \times \vec{u}}$

$\Rightarrow \vec{\ell}_1 = \vec{\ell}_2 = \vec{r} = \vec{a} + \dfrac{\vec{a} \cdot (\vec{b} \times \vec{v})}{\vec{v} \cdot (\vec{b} \times \vec{u})} \vec{u}$ 4

$\Rightarrow \underbrace{\vec{v} \cdot (\vec{b} \times \vec{u}) \vec{r}}_{\vec{c}} = \underbrace{\vec{v} \cdot (\vec{b} \times \vec{u})}_{p} \underbrace{\vec{a}}_{\vec{x}} + \vec{a} \cdot \underbrace{(\vec{b} \times \vec{v})}_{\vec{b}} \underbrace{\vec{u}}_{\vec{a}}$

$p + \vec{a} \cdot \vec{b} = 0 \iff \vec{v} \cdot (\vec{b} \times \vec{u}) + \vec{u} \cdot (\vec{b} \times \vec{v}) = 0$

$\Rightarrow$ solution from lecture

$\vec{x} = \dfrac{1}{p} \vec{c} + \kappa \vec{a} \qquad \kappa \in \mathbb{R}$

$\Rightarrow \quad \vec{a} = \dfrac{1}{\vec{v} \cdot (\vec{b} \times \vec{u})} \vec{v} \cdot (\vec{b} \times \vec{u}) \vec{r} + \kappa \vec{u}$

$\vec{a} = \vec{r} + \kappa \vec{u}$ 8

$\boxed{\Sigma = 15}$