Geometry & Vectors CW 2 Solutions

1) i) $p\vec{x} + (\vec{x}\cdot\vec{b})\vec{a} = \vec{c}$ $\qquad |\cdot\vec{b}$ $\qquad$ (1)

$\Rightarrow \quad p\,\vec{x}\cdot\vec{b} + (\vec{x}\cdot\vec{b})(\vec{a}\cdot\vec{b}) = \vec{c}\cdot\vec{b}$

$\Rightarrow \quad \vec{x}\cdot\vec{b} = \dfrac{\vec{c}\cdot\vec{b}}{p+\vec{a}\cdot\vec{b}}$ $\qquad$ for $p + \vec{a}\cdot\vec{b} \neq 0$

$\Rightarrow$ into (1) $\quad p\vec{x} + \dfrac{\vec{c}\cdot\vec{b}}{p+\vec{a}\cdot\vec{b}}\,\vec{a} = \vec{c} \Rightarrow \vec{x} = \dfrac{1}{p}\left(\vec{c} - \dfrac{\vec{c}\cdot\vec{b}}{p+\vec{a}\cdot\vec{b}}\right)$

When $p + \vec{a}\cdot\vec{b} = 0 \Rightarrow \vec{c}\cdot\vec{b} = 0 \Rightarrow \vec{x} = \dfrac{1}{p}\vec{c} + K\vec{a} \quad \forall K \in \mathbb{C}$

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ii) $\vec{a}\times|$ $\qquad \vec{x}\times\vec{a} = \vec{b}$

$\Rightarrow \quad \vec{a}\times\vec{x}\times\vec{a} = \vec{a}\times\vec{b}$

from lecture formula TP2: $\vec{u}\times\vec{v}\times\vec{w} = (\vec{u}\cdot\vec{w})\vec{v} - (\vec{u}\cdot\vec{v})\vec{w}$

$\Rightarrow \quad \vec{a}\times\vec{x}\times\vec{a} = (\vec{a}\cdot\vec{a})\vec{x} - (\vec{a}\cdot\vec{x})\vec{a} = \vec{a}\times\vec{b}$

This is of the form (1) with $p \equiv \vec{a}\cdot\vec{a}$, $\vec{b} \equiv -\vec{a}$, $\vec{c} \equiv \vec{a}\times\vec{b}$

$\Rightarrow \quad \vec{x} = \dfrac{1}{a^2}\vec{a}\times\vec{b} + K\vec{a}$ $\quad \because \quad p + \vec{a}\cdot\vec{b} \equiv \vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{a} = 0$

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iii) (1) $\quad p\vec{x} + q\vec{y} = \vec{a}$ $\qquad$ (2) $\quad \vec{x}\times\vec{y} = \vec{b}$

$\vec{x}\times(1)$: $\qquad p\underbrace{\vec{x}\times\vec{x}}_{0} + q\underbrace{\vec{x}\times\vec{y}}_{\vec{b}} = \vec{x}\times\vec{a}$

This is of the form $\vec{x}\times\vec{a} = \vec{b}$ as in ii) with $\vec{b} \to q\vec{b}$

$\Rightarrow \quad \vec{x} = \dfrac{q}{a^2}\vec{a}\times\vec{b} + K\vec{a}$

$\Rightarrow \quad \vec{y} = \dfrac{1}{q}\left(\vec{a} - \dfrac{pq}{a^2}\vec{a}\times\vec{b} - K\vec{a}\right) = \left(\dfrac{1}{q} - K\right)\vec{a} - \dfrac{p}{a^2}\vec{a}\times\vec{b}$

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$\boxed{\Sigma = 20}$

2/ We have to solve $y^2 = 4ax \qquad y = mx + c$

$$\Rightarrow \quad (mx+c)^2 = m^2x^2 + 2mcx + c^2 = 4ax$$

$$\Leftrightarrow \quad \underbrace{m^2}_{\alpha}x^2 + \underbrace{(2mc-4a)}_{\beta}x + \underbrace{c^2}_{\gamma} = 0$$

$$\Rightarrow \quad x_{1/2} = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$

To intersect only in one point we need $x_1 = x_2 \Leftrightarrow \beta^2 = 4\alpha\gamma$

$$\Rightarrow \quad 4m^2c^2 + 16a^2 - 16mca = 0 \quad \Rightarrow \quad \underline{c = a/m} \qquad \text{for } m \neq 0$$

At $P(x_0, y_0)$: $\quad y_0 = m x_0 + \frac{a}{m} \quad \Leftrightarrow \quad m^2 x_0 - m y_0 + a = 0$

$$\Rightarrow \quad m = y_0 / 2x_0$$

$$\Rightarrow \quad \underline{y_0 = \frac{y_0}{2x_0}x + \frac{2x_0}{y_0}a}$$

Evidently when $m = 0$ we have infinitely many solutions $y = c$, which are of course not tangents.

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$\boxed{\Sigma = 10}$

3) Parabola $\mathcal{P}$: $\quad y = 2x^2$

Circle $\mathcal{C}$: $\quad x^2 + (y-a)^2 = 1$

Find the points $P_\pm = \mathcal{P} \cap \mathcal{C}$

tangent: $\quad \frac{dy}{dx} = 4x \qquad 2x + 2(y-a)\frac{dy}{dx} = 0$

$$\Rightarrow \quad 2x + 2(y-a)4x = 0 \quad \Rightarrow \quad (y-a) = \frac{1}{4}$$

$$\Rightarrow \text{ into } \mathcal{C} \quad x^2 + \frac{1}{4^2} = 1 \quad \Rightarrow \quad \underline{x} = \pm\sqrt{\frac{15}{16}} = \underline{\pm\frac{\sqrt{15}}{4}}$$

$$\Rightarrow \text{ into } \mathcal{P} \quad \underline{y = 2\left(\frac{\sqrt{15}}{4}\right)^2 = \frac{15}{8}}$$

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$$\Rightarrow \quad \underline{P_\pm = \left(\pm\frac{\sqrt{15}}{4}, \frac{15}{8}\right) = \mathcal{P} \cap \mathcal{C}}$$

$\boxed{\Sigma = 10}$

4) $\overrightarrow{AB} = \vec{i} + 3\vec{j} + 4\vec{k}$ $\quad Q(\lambda+3,\ 3\lambda+2,\ 4\lambda+1) \in \overleftrightarrow{AB}$

$\overrightarrow{CD} = 2\vec{i} + 3\vec{j} + \vec{k}$ $\quad P(2\mu+4,\ 3\mu+2,\ -2+\mu) \in \overleftrightarrow{CD}$

$\overrightarrow{AD} = 3\vec{i} + 3\vec{j} - 2\vec{k}$ $\quad R(3\tilde{\lambda}+3,\ 3\tilde{\lambda}+2,\ -2\tilde{\lambda}+1) \in \overleftrightarrow{AD}$

$\overrightarrow{BC} = -3\vec{j} - 7\vec{k}$ $\quad S(4,\ -3\tilde{\mu}+5,\ -7\tilde{\mu}+5) \in \overleftrightarrow{BC}$

P = Q:

$$\left.\begin{array}{ll}(1) & \lambda+3 = 2\mu+4 \\ (2) & 3\lambda+2 = 3\mu+2\end{array}\right\} \quad 7 = 3\mu+10 \Rightarrow \mu = -1$$

into (2): $3\lambda+2 = -3+2 \Rightarrow \lambda=$

(3) $4\lambda+1 = \mu-2$

(3): $-4+1 = -1-2$ ✓

$\Rightarrow$ $Q = P(2, -1, -3)$

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R = S:

(1) $3\lambda+3 = 4 \Rightarrow \lambda = \frac{1}{3}$

(2) $3\lambda+2 = -3\mu+5 \Rightarrow 1+2 = -3\mu+5 \Rightarrow \mu=$

(3) $-2\lambda+1 = -7\mu+5 \Rightarrow -\frac{2}{3}+1 = -\frac{7\,2}{3}+5$ ✓

$\Rightarrow$ $R = S(4, 3, \frac{1}{3})$

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$\boxed{\Sigma = 10}$