# Solution coursework 2

1) For the parabola and the line to intersect we have to solve

$$y^2 = 4a x \qquad y = m x + c$$

$\Rightarrow \quad m^2 x^2 + 2 m c x + c^2 = 4 a x$

$\Leftrightarrow \quad \underbrace{m^2}_{\alpha} x^2 + \underbrace{(2mc - 4a)}_{\beta} x + \underbrace{c^2}_{\gamma} = 0$

$\Rightarrow \quad x_{1/2} = \dfrac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$ for a tangent we require $x_1 = x_2$

$\Rightarrow \quad \beta^2 = 4\alpha\gamma \quad \Leftrightarrow \quad 4m^2c^2 + 16a^2 - 16mca = 0 \quad \Rightarrow \quad \underline{c = a/m}$

At the point $P(x_0, y_0)$: $\quad y_0 = m x_0 + \frac{a}{m} \quad \Leftrightarrow \quad m^2 x_0 - m y_0 + a = 0$

$\Rightarrow \quad \underline{m = y/2x_0}$

$\Rightarrow \quad \underline{y = \dfrac{y_0}{2x_0} x + \dfrac{2x_0}{y_0} a}$ is the equation of the tangent! $\boxed{10}$

2)

$$2x^2 + 3y^2 - 4x + 5y + 4 = 0$$

Complete the square:

$$2(x^2 - 2x + 2) - 2 + 3\left(y^2 + \tfrac{5}{3}y + \left(\tfrac{5}{6}\right)^2\right) - \left(\tfrac{5}{6}\right)^2 3 + 4 = 0$$

$\Leftrightarrow \quad 2(x-1)^2 + 3(y + 5/6)^2 = 2 - 4 + \frac{25}{12} = \frac{1}{12}$

$\Rightarrow$ The normal form of the ellipse is:

$$\underline{\frac{(x-1)^2}{1/24} + \frac{(y + \frac{5}{6})^2}{1/36} = 1} \quad \Rightarrow \quad a^2 = \frac{1}{24}, \quad b^2 = \frac{1}{36}$$

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$\Rightarrow$ the centre is at $(1, -\frac{5}{6})$ 1

$\Rightarrow$ the length of the major axis is $2a = 2/\sqrt{24} = \frac{1}{\sqrt{6}}$

$\Rightarrow$ the length of the minor axis is $2b = 2/\sqrt{36} = \frac{1}{3}$ 1

$\Rightarrow$ eccentricity is $e = \sqrt{1 - b^2/a^2} = \sqrt{1 - 24/36} = \sqrt{1 - 2/3} = 1/\sqrt{3}$ 1

$\Rightarrow$ the foci are at $(1 \pm e a, -5/6) = \left(1 \pm \frac{1}{\sqrt{3 \cdot 24}}, -\frac{5}{6}\right)$ 1

$\Rightarrow$ the vertices are at $(1 \pm a, -\frac{5}{6}) = \left(1 \pm \frac{1}{2\sqrt{6}}, -5/6\right)$ 1

$\Rightarrow$ the equation of the directrix is $x = 1 - \frac{k}{e} = 1 - \frac{b^2}{ae} = 1 - \frac{1}{36}\, 2\cdot\sqrt{6}\sqrt{3} = 1 - \frac{\sqrt{2}}{6}$

$\boxed{10}$

3) $A(3,1,2) \qquad B(-1,2,0) \quad \Rightarrow \quad \vec{AB} = -4\vec{i} + \vec{j} - 2\vec{k}$

i) Equation of the line through A and B:

$$\mathcal{L}: \frac{x-3}{-4} = \frac{y-1}{1} = \frac{z-2}{-2}$$

$\Rightarrow P(-4\lambda + 3, \lambda + 1, -2\lambda + 2) \in \mathcal{L}$

$\Rightarrow P \in xz$-plane for $y = 0 \quad \Rightarrow$ for $\lambda = -1$

$\Rightarrow$ The point of intersection is $\underline{(7, 0, 4)}$. (5)

ii) The line intersects the plane $\mathcal{P}: 2x - 4y - z = 5$ when

$P(-4\lambda+3, \lambda+1, -2\lambda+2) \in \mathcal{P} \Leftrightarrow -8\lambda + 6 - 4\lambda - 4 + 2\lambda - 2 = 5$

$\Rightarrow \quad -10\lambda = 5 \quad \Rightarrow \quad \underline{\lambda = -\frac{1}{2}}$

$\Rightarrow \quad \underline{\mathcal{L} \cap \mathcal{P} = Q(5, \tfrac{1}{2}, 3)}$ (5) $\boxed{10}$

4) $\mathcal{L}_1: \frac{x-3}{1} = \frac{y-1}{2} = \frac{z+1}{3} = \lambda \qquad \mathcal{L}_2: \frac{x-2}{1} = \frac{y-5}{-1} = \frac{z}{1} = \mu$

i) $\mathcal{L}_1$ and $\mathcal{L}_2$ intersect iff:

$$P(\lambda+3, 2\lambda+1, 3\lambda-1) = Q(\mu+2, 5-\mu, \mu)$$

$\Rightarrow$ (1) $\lambda + 3 = \mu + 2$ $\quad \overset{(1)+(3)}{\Rightarrow} \lambda + 3 = 3\lambda - 1 + 2 \quad \overset{(1)}{\Rightarrow} \lambda = 1 \quad \Rightarrow \mu = 2$

(2) $2\lambda + 1 = 5 - \mu$

(3) $3\lambda - 1 = \mu$ $\qquad$ verify (2) LHS: $2\cdot 1 + 1 = 3$ RHS: $5 - 2 = 3$ ✓

$\Rightarrow \quad \underline{P(4,3,2) = \mathcal{L}_1 \cap \mathcal{L}_2}$ (5)

ii) $\mathcal{L}_1$ is parallel to the vector $\vec{V_1} = \vec{i} + 2\vec{j} + 3\vec{k}$

$\mathcal{L}_2$ is parallel to the vector $\vec{V_2} = \vec{i} - \vec{j} + \vec{k}$

$\Rightarrow \vec{V_1} \times \vec{V_2}$ is perpendicular to $\mathcal{L}_1$ and $\mathcal{L}_2$

$$\Rightarrow \quad \vec{V}_1 \times \vec{V}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 1 & -1 & 1 \end{vmatrix} = 5\vec{i} + 2\vec{j} - 3\vec{k}$$

$\Rightarrow \quad \mathcal{P}: \quad 5x + 2y - 3z = d$

$\Rightarrow \quad Q \in \mathcal{L}_2 \quad \therefore \quad Q \in \mathcal{P}$ take $\mu = 0 \Rightarrow (2, 5, 0) \in \mathcal{P}$

$\Rightarrow \quad 5 \cdot 2 + 2 \cdot 5 = d \quad \Rightarrow \quad \underline{d = 20}$

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$\Rightarrow \quad \underline{\mathcal{P}: \; 5x + 2y - 3z = 20}$

$\boxed{10}$

5) A normal vector to $\mathcal{P}_1$: $\vec{n}_1 = 3\vec{i} - \vec{j} + \vec{k}$

a normal vector to $\mathcal{P}_2$: $\vec{n}_2 = -\vec{i} + 2\vec{j} + 6\vec{k}$

$\Rightarrow \quad \vec{n}_1 \times \vec{n}_2$ is parallel to $\mathcal{L} = \mathcal{P}_1 \cap \mathcal{P}_2$

$$\Rightarrow \quad \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 1 \\ -1 & 2 & 6 \end{vmatrix} = \vec{i}\begin{vmatrix} -1 & 1 \\ 2 & 6 \end{vmatrix} - \vec{j}\begin{vmatrix} 3 & 1 \\ -1 & 6 \end{vmatrix} + \vec{k}\begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= -8\vec{i} - 19\vec{j} + 5\vec{k}$$

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Any point on $\mathcal{L}$ satisfies $(z = 0)$

$$\left.\begin{aligned} 3x - y &= 11 \\ -x + 2y &= -7 \end{aligned}\right\} \Rightarrow 5x = 15 \Rightarrow \underline{x = 3} \Rightarrow \underline{y = -2}$$

$$\Rightarrow \quad \mathcal{L}: \quad \frac{x-3}{-8} = \frac{y+2}{-19} = \frac{z}{5} = \lambda$$

is the line of intersection.

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$\boxed{10}$