Coursework 2

1) see lecture notes p. 34-35   (30)

2) i) the major axis is along the Y-axis:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

$P(1,0) \in$ ellipse $\Rightarrow \frac{1}{b^2} = 1 \Rightarrow \underline{b^2 = 1}$

the foci are at $\pm ae = \pm a\sqrt{1 - b^2/a^2} = \pm 2 \Rightarrow a^2\left(1 - \frac{1}{a^2}\right) = 4 \Rightarrow \underline{a^2 = 5}$

$\boxed{8}$ $$\Rightarrow \underline{\underline{x^2 + \frac{y^2}{5} = 1}}$$

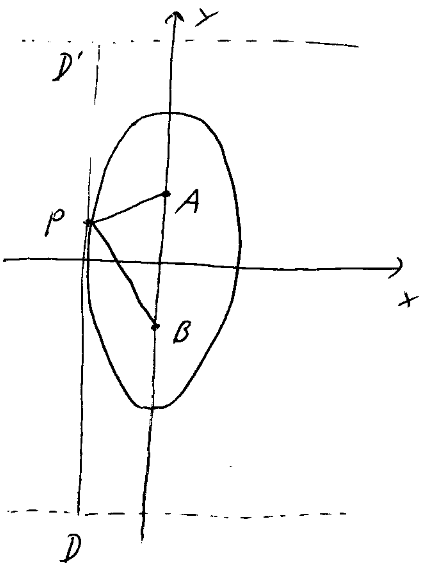
ii) the asymptotes are at $bx \mp ay = 0 \iff y = \pm \frac{b}{a} x = \pm 2x$

$\Rightarrow \underline{b/a = 2}$

the directrices are at $x = \pm a/e = \pm 1 \Rightarrow a = e = \sqrt{1 + b^2/a^2}$

$\boxed{8}$ $\Rightarrow \underline{a = \sqrt{5}} \Rightarrow \underline{b = 2 \cdot \sqrt{5}} \Rightarrow \underline{\underline{\frac{x^2}{5} - \frac{y^2}{20} = 1}}$

iii)



$|BP| = e\,|PD|$
$|AP| = e\,|PD'|$ per definition

$\Rightarrow |AP| + |BP| = e\left(|PD| + |PD'|\right) = e\,|DD'|$

$= 2\frac{a}{e} \cdot e = 2a = 7$

$\Rightarrow \underline{a = \frac{7}{2}}$

$\boxed{9}$

the foci are at $\pm ea = \pm 3 \Rightarrow \frac{6}{7} = \sqrt{1 - \frac{b^2}{a^2}} \Rightarrow \frac{36}{49} = 1 - \frac{b^2}{a^2}$

(25) $\Rightarrow \underline{b^2 = \frac{13}{4}} \Rightarrow \underline{\underline{\frac{x^2}{\left(\frac{13}{4}\right)} + \frac{y^2}{\left(\frac{49}{4}\right)} = 1}}$

3) The equation of the tangent at the point $Q(x_1, y_1)$ is:

$\mathcal{L}: \frac{x x_1}{36} + \frac{y y_1}{20} = 1 \qquad P(8,0) \in \mathcal{L} \Rightarrow \frac{8x_1}{36} = 1 \Rightarrow \underline{x_1 = \frac{9}{2}}$

$Q \in \mathcal{L} \Rightarrow \frac{x_1^2}{36} + \frac{y_1^2}{20} = 1 \Rightarrow \frac{y_1^2}{20} = 1 - \frac{\cancel{81}\cdot 9}{4\cdot \cancel{36}\, 4} \Rightarrow y_1^2 = 20 - \frac{20\cdot 9}{4\cdot 4}$

15 $\Rightarrow \underline{y_1 = \pm \frac{\sqrt{35}}{2}}$

$\Rightarrow \mathcal{L}: \quad \underline{\underline{y}} = \frac{20}{y_1}\left(1 - \frac{x_1 x}{36}\right) = \pm \frac{20\cdot 2}{\sqrt{35}}\left(1 - \frac{9}{36\cdot 2}x\right) = \underline{\underline{\pm\sqrt{\frac{5}{7}}\,(8-x)}}$

4) The vectors $\vec{BA} = 2\vec{i} + \vec{k}$ and $\vec{CA} = 2\vec{i} + \vec{j} - 2\vec{k}$ are in the plane

$\Rightarrow \vec{BA} \times \vec{CA} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 1 \\ 2 & 1 & -2 \end{vmatrix} = -\vec{i} - \vec{j}\begin{vmatrix} 2 & 1 \\ 2 & -2 \end{vmatrix} + k\begin{vmatrix} 2 & 0 \\ 2 & 1 \end{vmatrix} = -\vec{i} + 6\vec{j} + 2\vec{k}$

is perpendicular to the plane

- take $\bar{X}(x, y, z)$ in the plane

$\Rightarrow \vec{AX} = (x-3)\vec{i} + (y-1)\vec{j} + (z-1)\vec{k}$ is in the plane

$\Rightarrow \vec{AX} \cdot (\vec{BA} \times \vec{CA}) = 0 = (3-x) + (6y-6) + (2z-2) = 0$

$\boxed{7}$ $\Rightarrow \underline{\underline{\mathcal{P}: \quad x - 6y - 2z + 5 = 0}}$

- Any point Q on the line $\mathcal{L}$ is of the form $Q(2\lambda+1, 2\lambda, -3\lambda+1)$

$\boxed{3}$ $\Rightarrow Q \in \mathcal{P} \Leftrightarrow (2\lambda+1) - 6(2\lambda) - 2(1-3\lambda) + 5 = 0$

$\Leftrightarrow 2\lambda - 12\lambda + 6\lambda + 1 - 2 + 5 = 0 \Rightarrow \underline{\lambda = 1}$

10 $\Rightarrow \underline{\underline{Q(3, 2, -2) \in \mathcal{P} \text{ and } Q \in \mathcal{L}}}$

5) Normal vector to $\mathcal{P}_1$: $\vec{n}_1 = 5\vec{i} + 4\vec{j} + 7\vec{k}$

normal vector to $\mathcal{P}_2$: $\vec{n}_2 = 2\vec{i} + 3\vec{j} + 2\vec{k}$

$\Rightarrow \vec{n}_1 \times \vec{n}_2$ is parallel to $\mathcal{L} = \mathcal{P}_1 \cap \mathcal{P}_2$

$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & 4 & 7 \\ 2 & 3 & 2 \end{vmatrix} = \vec{i}\begin{vmatrix} 4 & 7 \\ 3 & 2 \end{vmatrix} - \vec{j}\begin{vmatrix} 5 & 7 \\ 2 & 2 \end{vmatrix} + \vec{k}\begin{vmatrix} 5 & 4 \\ 2 & 3 \end{vmatrix} = -13\vec{i} + 4\vec{j} + 7\vec{k}$

- Any point $P(x_1, y_1)$ on $\mathcal{L}$ has to satisfy $5x_1 + 4y_1 = 26$

$2x_1 + 3y_1 = 11$

$\Rightarrow 3x_1 + y_1 = 15 \quad \Rightarrow \quad y_1 = 15 - 3x_1$

$\Rightarrow 2x_1 + 45 - 9x = 11 \quad \Rightarrow \quad x_1 = \frac{34}{7} \quad \Rightarrow \quad y_1 = \frac{3}{7}$

10 $\quad \Rightarrow \mathcal{L}: \quad \dfrac{x - \frac{34}{7}}{-13} = \dfrac{y - \frac{3}{7}}{4} = \dfrac{z}{7} = \lambda$

6) $\left.\begin{array}{l} \vec{v} = 2\vec{i} + 3\vec{j} - 2\vec{k} \\ \vec{w} = 3\vec{i} + 2\vec{j} + \vec{k} \end{array}\right\}$ are parallel to the plane

$\Rightarrow \vec{v} \times \vec{w}$ is perpendicular to the plane

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -2 \\ 3 & 2 & 1 \end{vmatrix} = \vec{i}\begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & -2 \\ 3 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = 7\vec{i} - 8\vec{j} - 5\vec{k}$$

$\Rightarrow \quad \mathcal{P}: \quad 7x - 8y - 5z + d = 0$

$P(2,1,1) \in \mathcal{L}_1 \quad$ and $\quad P(2,1,1) \in \mathcal{P}$

$\Rightarrow \quad 7 \cdot 2 - 8 - 5 + d = 0 \quad \Rightarrow \quad d = -1$

10 $\quad \Rightarrow \quad \mathcal{P}: \; 7x - 8y - 5z - 1 = 0$