Geometry & Vectors

Solutions to coursework 1

1) Taken from http://mathworld.wolfram.com/

Axiom: A proposition regarded as self-evidently true without proof. The word "axiom" is a slightly archaic synonym for postulate. Compare conjecture or hypothesis, both of which connote apparently true but not self-evident statements.

Proposition: A statement which is to be proved.

Theorem: A theorem is a statement that can be demonstrated to be true by accepted mathematical operations and arguments. In general, a theorem is an embodiment of some general principle that makes it part of a larger theory. The process of showing a theorem to be correct is called a proof.

Although not absolutely standard, the Greeks distinguished between "problems" (roughly, the construction of various figures) and "theorems" (establishing the properties of said figures; Heath 1956, pp. 252, 262, and 264).

According to the Nobel Prize-winning physicist Richard Feynman, any theorem, no matter how difficult to prove in the first place, is viewed as "trivial" by mathematicians once it has been proven. Therefore, there are exactly two types of mathematical objects: trivial ones, and those which have not yet been proven.

The late mathematician P. Erdos described a mathematician as "a machine for turning coffee into theorems" (Hoffman 1998, p. 7). R. Graham has estimated that upwards of 250000 mathematical theorems are published each year (Hoffman 1998, p. 204).

Lemma: A short theorem used in proving a larger theorem.

Corollary: An immediate consequence of a result already proved. Corollaries usually state more complicated theorems in a language simpler to use and apply.

2) $\vec{u} = -\vec{i} + \lambda\vec{j} + \vec{k}$ $\quad\quad$ $\vec{v} = 2\vec{j} - \vec{k}$

i)

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & \lambda & 1 \\ 0 & 2 & -1 \end{vmatrix} = -(\lambda+2)\vec{i} - \vec{j} - 2\vec{k} =: \vec{w}$$

$\Rightarrow$ The vectors $\pm\vec{w}$ are perpendicular to the plane which contains $\vec{u}$,

$|\vec{w}|^2 = 9 = (\lambda+2)^2 + 1 + 4 \quad \Rightarrow \quad \underline{\lambda = 0}$ and $\underline{\lambda = -4}$

ii) $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$ $\quad$ $\vec{u}\cdot\vec{v} = 2\lambda - 1$; $|\vec{u}| = \sqrt{2+\lambda^2}$; $|\vec{v}| = \sqrt{5}$

$$\Rightarrow \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} = \frac{2\lambda-1}{\sqrt{5}\sqrt{\lambda^2+2}} \quad \Rightarrow \quad \frac{3}{4} = \frac{(2\lambda-1)^2}{5\cdot(\lambda^2+2)}$$

$$\Rightarrow \quad 0 = \lambda^2 - 16\lambda - 26 \quad \Rightarrow \quad \lambda_{1/2} = 8 \pm 3\sqrt{10}$$

Discard the minus sign, since $\dfrac{2\lambda-1}{\sqrt{5}\sqrt{\lambda^2+2}} > 0$.

iii) for $\lambda = 1 \Rightarrow$ $\quad \vec{u}\cdot\vec{v} = 1 \quad |\vec{u}| = \sqrt{3} \quad |\vec{v}| = \sqrt{5}$

CS: $\quad -\sqrt{3}\sqrt{5} \le 1 \le \sqrt{3}\sqrt{5}$

$\quad\quad -\sqrt{15} \le 1 \le \sqrt{15}$ ✓

TI: $\quad \vec{u} + \vec{v} = -\vec{i} + 3\vec{j}$

$$\Rightarrow \quad |\vec{u}+\vec{v}| = \sqrt{10} \simeq 3.16 \le |\vec{u}| + |\vec{v}| = \sqrt{3} + \sqrt{5} \simeq 3.97$$

The equal signs hold iff $\vec{u} = \gamma\vec{v}$, but no such γ exists.

3) i) The vector $\vec{u}\times\vec{v}$ is perpendicular to $\vec{u}$ and $\vec{v}$.

$\Rightarrow \quad (\vec{u}\times\vec{v})\cdot\vec{u} = |\vec{u}\times\vec{v}|\,|\vec{u}|\cos\frac{\pi}{2} = 0$ $\;\Big|\;$ or $(\vec{u}\times\vec{v})\cdot\vec{u} = -\underbrace{(\vec{u}\times\vec{u})}_{0}\cdot\vec{v} =$

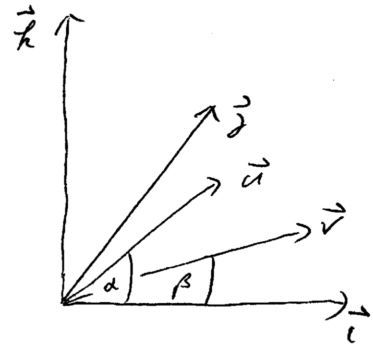
ii) $(\vec{u}+\vec{v})\cdot(\vec{v}+\vec{w})\times(\vec{w}+\vec{u})$

$$= (\vec{u}+\vec{v})\cdot[\vec{v}\times\vec{w} + \vec{v}\times\vec{u} + \underbrace{\vec{w}\times\vec{w}}_{0} + \vec{w}\times\vec{u}]$$

$$= \vec{u}\cdot(\vec{v}\times\vec{w}) + \underbrace{\vec{u}\cdot(\vec{v}\times\vec{u})}_{0} + \underbrace{\vec{u}\cdot(\vec{w}\times\vec{u})}_{0} + \underbrace{\vec{v}\cdot(\vec{v}\times\vec{w})}_{0} + \underbrace{\vec{v}\cdot(\vec{v}\times\vec{u})}_{0} + \vec{v}\cdot(\vec{w}\times\vec{u}$$

$$= \vec{u}\cdot(\vec{v}\times\vec{w}) - \vec{u}\cdot(\vec{w}\times\vec{v}) = 2\,\vec{u}\cdot(\vec{v}\times\vec{w})$$

4)

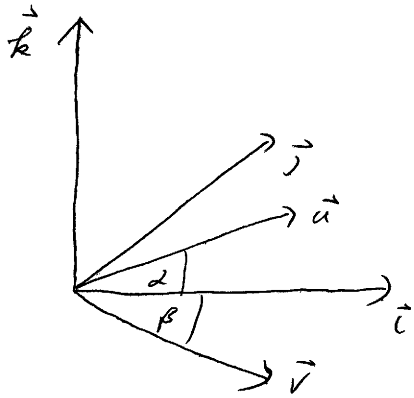


$$\vec{u} = \cos\alpha\,\vec{i} + \sin\alpha\,\vec{j}$$

$$\vec{v} = \cos\beta\,\vec{i} + \sin\beta\,\vec{j}$$

$$\Rightarrow \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \end{vmatrix} = \vec{k}\,(\cos\alpha\sin\beta - \sin\alpha\cos\beta$$

$$\vec{v} \times \vec{u} = |\vec{v}|\,|\vec{u}|\,\sin(\alpha-\beta)\;\vec{k} \Rightarrow \underline{\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta}$$



$$\vec{u} = \cos\alpha\,\vec{i} + \sin\alpha\,\vec{j}$$

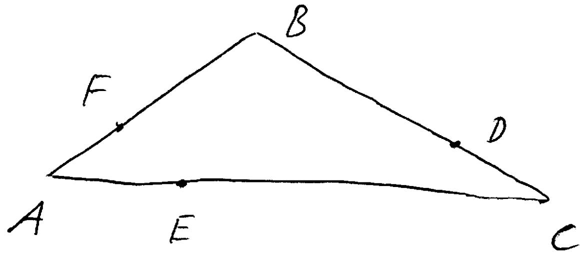
$$\vec{v} = \cos\beta\,\vec{i} - \sin\beta\,\vec{j}$$

$$\Rightarrow \vec{u} \times \vec{v} = -\vec{k}\,(\cos\alpha\sin\beta + \sin\alpha\cos\beta)$$

$$\Rightarrow \vec{v} \times \vec{u} = |\vec{v}|\,|\vec{u}|\,\sin(\alpha+\beta)$$

$$\Rightarrow \underline{\sin(\alpha+\beta) = \cos\alpha\sin\beta + \cos\alpha\sin\beta}$$

5)



$$\overrightarrow{AD} = \overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AC} + \frac{1}{3}\overrightarrow{CB}$$

$$= \overrightarrow{AC} + \frac{1}{3}\left(\overrightarrow{CA} + \overrightarrow{AB}\right)$$

$$= \frac{2}{3}\overrightarrow{AC} + \frac{1}{3}\overrightarrow{AB}$$

$$\left.\begin{aligned} \overrightarrow{EB} &= \overrightarrow{EA} + \overrightarrow{AB} = \overrightarrow{AB} - \frac{1}{3}\overrightarrow{AC} \\ \overrightarrow{FC} &= \overrightarrow{FA} + \overrightarrow{AC} = \overrightarrow{AC} - \frac{2}{3}AB \end{aligned}\right\} \Rightarrow \overrightarrow{EB} + \overrightarrow{FC} = \frac{2}{3}\overrightarrow{AC} + \frac{1}{3}\overrightarrow{AB}$$

$$\Rightarrow \underline{\overrightarrow{AD} = \overrightarrow{EB} + \overrightarrow{FC}}$$