1) Axioms for geometry

1.1) What is geometry?

www.staff.city.ac.uk/~fring/Geo&Vect/index.html

Some history:

- The first people to do geometry were the ancient Egyptians (1300BC). The main purpose was to determine the size of land (washed away by the Nile → tax reduction). From this originates the meaning of the word geometry ≡ "earth measurement".
- The Greeks were the first to turn geometry into Mathematics.
  - i) They made geometry abstract, that is they invented idealised notions of lines, points, planes, ...
  - ii) They made geometry deductive: That is from some "unquestionable" premisses called axioms (postulates) one derives some laws by using logical reasoning.

⇒ The most successful scientific textbook ever written: EUCLID'S "Elements" (300BC)

(From 5 Axioms he deduced the entire geometry know at the time)

For hundreds of years this book was considered as the supreme example of human reasoning. It contained the model for the entire mathematics.

Axiomatic system

$\boxed{\text{undefined terms}}$ (points, lines, planes, ...) + $\boxed{\text{Axioms}}$ (basic assumptions

⇓ logic

$\boxed{\text{defined terms}}$ (square, ...) $\boxed{\text{Theorems}}$ (propositions, corollaris, lemmas)

1.2) Some literature

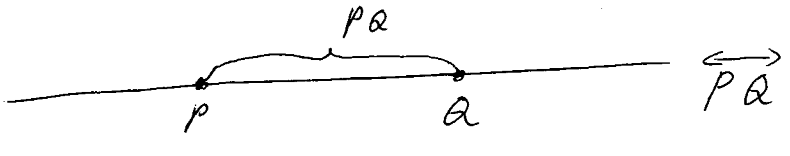
- Euclid's Elements
- Vector Analysis, M. R. Spiegel (McGraw-Hill, 1974)
- Advanced Calculus, M. R. Spiegel (McGraw-Hill, 1988)
- Calculus and Analytic geometry, R. L. Finney and G. B. Thomas (Addison-Wesley, 1994)
- Pure Mathematics 2, L. Bostock and S. Chandler (Thornes, 1990)
- Elementary Geometry, J. Roe. (Oxford Uni. Press, 1993)
- E. A. Abbott, Flatland (1884) (Perseus Press, 2003)
- I. Stewart, Flatterland (MacMillan, 2001)

1.3) The axioms of incidence

We leave some objects undefined:
points P, lines $\mathcal{L}$, planes $\mathcal{P}$, rulers, distance, area, volume,

Axiom 1 (Line axiom):

Through any two distinct points P and Q there is exactly one line $\mathcal{L} = \overleftrightarrow{PQ}$.



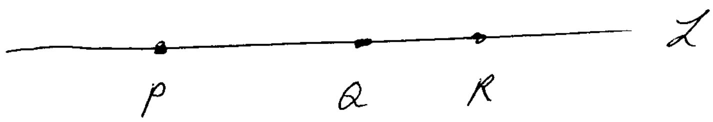
- PQ denotes the line segment and $\overleftrightarrow{PQ}$ denotes the line (entire) through P and Q.
- Even from this simple statement alone we can deduce something not so obvious.

Prop.: Two distinct lines L_1 and L_2 can cross at most in one point P.

Proof: Suppose L_1 and L_2 meet (cross, intersect, ...) in two points P_1 and P_2. This implies there are two distinct lines through P_1 and P_2, but this contradicts axiom 1.

Def.: Two lines which do not cross are <u>parallel</u>, if they are in the same plane and are said to be <u>skew</u>, if they are in different directions.

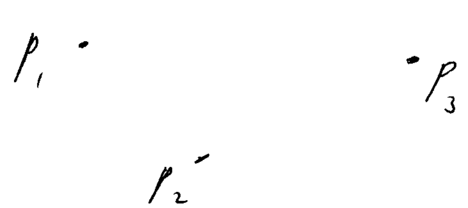
Def.: Three or more points are <u>collinear</u> if there is a line through all of them.



$P, Q, R \in L$

Axiom 2 (Plane Axiom):

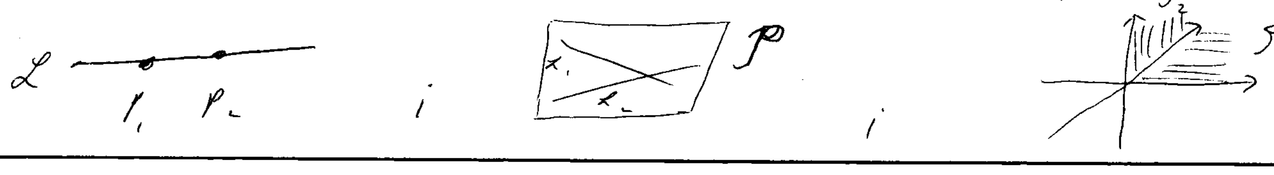
Through any three noncollinear points there is exactly one plane.



$P_1, P_2, P_3 \in \mathcal{P}$ = blackboard

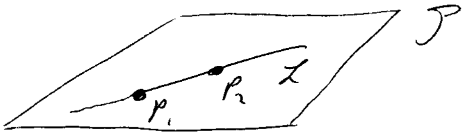
Axiom 3 (Dimension Axiom):

Any line contains at least two distinct points.
" plane " " " " " lines.
There are at least 2 distinct planes in space.



Axiom 4 (Line-Plane Intersection axiom)

If two distinct points of a line $\mathcal{L}$ lie in a plane $\mathcal{P}$, then the whole line lies in $\mathcal{P}$. $P_1, P_2 \in \mathcal{L}, \mathcal{P} \Rightarrow \mathcal{L} \in \mathcal{P}$



Prop.: Given a line $\mathcal{L}$ and a point $P \notin \mathcal{L}$, there is a unique plane $\mathcal{P}$ containing both $\mathcal{L}$ and P.

Proof: - from axiom 3 $\Rightarrow$ $\exists$ at least two distinct points $Q, R \in \mathcal{L}$

- P, Q, R are not collinear since the unique line $\mathcal{L}$ through Q and R does not pass through P
- axiom 2 $\Rightarrow$ there is a unique plane $\mathcal{P}$ through P, Q, R
- $\mathcal{P}$ contains two points of $\mathcal{L}$, i.e. $Q, R \Rightarrow \mathcal{L} \in \mathcal{P}$ by axiom 4

Def.: Two lines $\mathcal{L}_1, \mathcal{L}_2$ are coplanar if there is a plane $\mathcal{P}$ which contains both of them, i.e. $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{P}$.

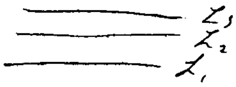
Axiom 5 (Parallel axiom)

For a given point P and line $\mathcal{L}$ there is one and only one line $\mathcal{L}'$ which passes through P and is parallel to $\mathcal{L}$

- controversial for 2000 years
- involves the notion of infinity
- replace it by something else $\Rightarrow$ non-Euclidean geometry

Here we accept it.

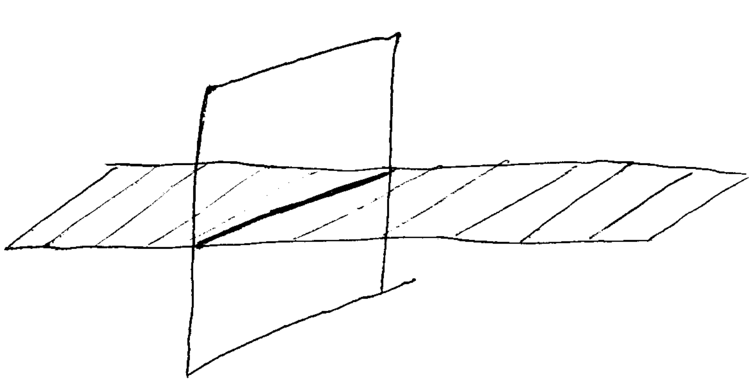
Prop.: Consider three coplanar lines L_1, L_2, L_3. If L_1 is parallel to L_2 and L_2 is parallel to L_3, then L_1 is parallel to L_3. (transitivity)



Proof:
- suppose that L_1 and L_3 are not parallel
- since the lines are coplanar, they have to cross each other at some point P if they are not parallel
- this contradicts axiom 5 as there are now two lines through P which are parallel to L_2

Axiom 6 (Plane - Plane intersection axiom)

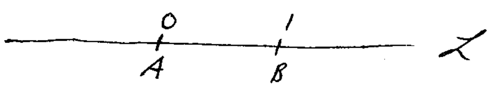
If two distinct planes meet, then their intersection is a line.



- So far we ~~have~~ no numbers. =)

1.4) Measurement axioms

We need a ruler on a line L, that is a one-to-one correspondence between $L \to \mathbb{R}$. Suppose one specifies the origin and a scale by some points A and B then all other points are uniquely determined.



Axiom 7 (ruler axiom):

Let L be a line through two distinct points A and B. Then there is a ruler on L in terms of which A corresponds to 0 and B corresponds to 1.

- We may change from one ruler to another ruler by means of a coordinate transformation. This is specified by

<u>Axiom 8 (rule comparison axiom)</u>

Two different rulers on the same line are related by the transformation

$$x' = U(x) = a x + b \qquad a, b \in \mathbb{R}.$$

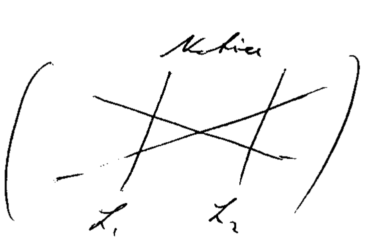
<u>Example</u>: Celsius to Fahrenheit temperature scale

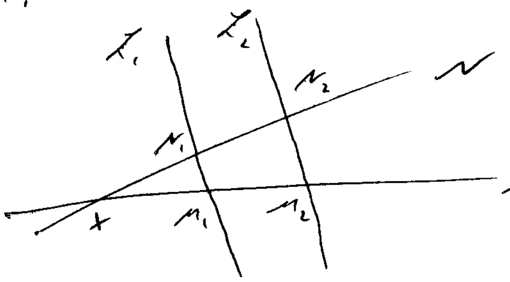
$$x' = \frac{9}{5} x + 32$$

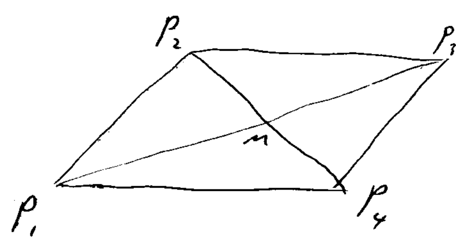
- Rulers enable us to talk about <u>ratios</u> of distances between points on a line. We did not assume that there is any absolute scale.

<u>Axiom 9 (similarity axiom)</u>

Let L_1 and L_2 be two parallel line in a plane P and M and N two different in the same plane crossing L_1 and L_2 in the points M_1, N_1, M_2, N_2 and intersecting at the point X. Then the ratios $XM_1 : XM_2$ and $XN_1 : XN_2$ are identical.





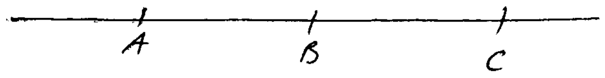


We want to define one more object:

<u>Def.</u>: Four points P_i for $i = 1, 2, 3, 4$ form a <u>parallelogram</u> if the midpoints of $P_1 P_3$ is equal to the midpoint of $P_2 P_4$.

where we did not yet say what we mean by midpoint:

Def.: The midpoint of the line segment AC is defined to be the unique point B on AC such that $AB : BC = 1 : 1$



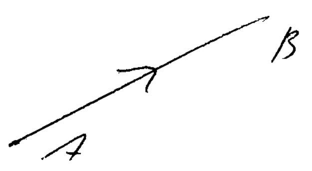
2) Vectors

2.1) The notion of a vector

There are scalar quantities in physics such as masses, length, volumes, temperatures, energies, ..., i.e. they are just numbers with some units attached (kg, m, C°, J ...). There are other quantities which besides being associated to numbers also have a direction attached to them, such as velocities, accelerations, forces, electric and magnetic fields, electric currents, etc.

Thus one may think of a vector as having a magnitude as well as a direction.

One can depict a vector as an arrow on a line segment



$\overrightarrow{AB}$ or $\underline{AB}$ or $\vec{AB} = \vec{v}$

Note that $\overrightarrow{AB} \neq \overleftarrow{AB}$!

Adding two vectors then corresponds to "shifting" arrows.