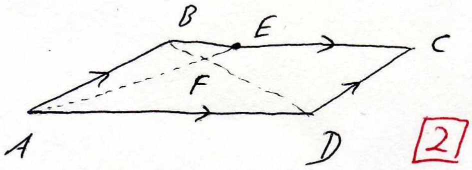
Geometry & Vectors (Exam 06)

1)



$\boxed{2}$

$$\vec{AE} = \vec{AB} + \tfrac{1}{3}\vec{BC} \Rightarrow \vec{AF} = \kappa\left(\vec{AB} + \tfrac{1}{3}\vec{AD}\right)$$

$$\vec{BD} = \vec{AD} - \vec{AB} \Rightarrow \vec{BF} = \lambda\left(\vec{AD} - \vec{AB}\right)$$

Also $\vec{AF} = \vec{AB} + \vec{BF} = \vec{AB} + \lambda(\vec{AD} - \vec{AB}) = \kappa(\vec{AB} + \tfrac{1}{3}\vec{AD})$

$$\Leftrightarrow \quad \vec{AB} + \lambda\vec{AD} - \lambda\vec{AB} = \kappa\vec{AB} + \frac{\kappa}{3}\vec{AD}$$

$$\Leftrightarrow \quad (1 - \lambda - \kappa)\vec{AB} = \left(\frac{\kappa}{3} - \lambda\right)\vec{AD}$$

$$\Rightarrow \quad \lambda + \kappa = 1 \quad \wedge \quad \lambda = \frac{\kappa}{3}$$

$$\Rightarrow \quad \frac{\kappa}{3} + \kappa = 1 \Rightarrow \underline{\kappa = \frac{3}{4}} \Rightarrow \underline{\lambda = \frac{1}{4}}$$

$$\Rightarrow \quad \underline{\underline{\vec{AF} = \frac{3}{4}\vec{AE}}} \qquad \boxed{6}$$

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2) $\vec{u} = 2\vec{i} + 2\vec{k} \qquad \vec{v} = 2\vec{i} - \vec{j} + \vec{k}$

i)

$$\left.\begin{aligned} &\vec{u}\cdot\vec{v} = 6 \\ &|\vec{u}| = \sqrt{4+4} = 2\sqrt{2} \\ &|\vec{v}| = \sqrt{4+1+1} = \sqrt{6} \end{aligned}\right\} \Rightarrow \cos\theta = \frac{6}{\sqrt{6}\sqrt{2}\,2} = \frac{\sqrt{3}}{2} \Rightarrow \underline{\underline{\theta = \frac{\pi}{6}}}$$

$\boxed{3}$

ii) Take a general vector of the form $\vec{w} = a\vec{i} + b\vec{j} + c\vec{k}$

$$\Rightarrow \vec{w}\cdot\vec{w} = a^2 + b^2 + c^2 = 3^2$$

$$\vec{w} \perp \vec{u} \Leftrightarrow \vec{u}\cdot\vec{w} = 0 = 2a + 2c \Rightarrow \underline{c = -a}$$

$$\vec{w} \perp \vec{v} \Leftrightarrow \vec{v}\cdot\vec{w} = 0 = 2a - b + c \Rightarrow a - b = 0 \Rightarrow \underline{a = b}$$

$$\Rightarrow 3^2 = a^2 + a^2 + a^2 \Rightarrow \underline{a = \pm\sqrt{3}} \Rightarrow \underline{\underline{\vec{w}_{1/2} = \pm\sqrt{3}(\vec{i} + \vec{j} - \vec{k})}} \ \boxed{3}$$

iii) The triangle inequality reads

$$|\vec{u} + \vec{v}| \le |\vec{u}| + |\vec{v}|$$

$$\vec{u} + \vec{v} = 4\vec{i} - \vec{j} + 3\vec{k} \Rightarrow |\vec{u} + \vec{v}| = \sqrt{16+1+9} = \sqrt{26}$$

$$\Rightarrow \quad \sqrt{26} \le 2\sqrt{2} + \sqrt{6}$$

$$\Leftrightarrow \quad 5.10 \le 2.83 + 2.45 = 5.28 \quad ✓ \qquad \boxed{2}$$

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3) $\vec{u} = \vec{j} + 2\vec{k}$ $\quad \vec{v} = -\vec{i} + 3\vec{j}$ $\quad \vec{w} = \vec{i} - \vec{j}$ $\quad \vec{x} = \vec{i} + \vec{j} - \vec{k}$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 2 \\ -1 & 3 & 0 \end{vmatrix} = -6\vec{i} - 2\vec{j} + \vec{k} \qquad \vec{w} \times \vec{x} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{vmatrix} = \vec{i} + \vec{j} + 2\vec{k}$$ $\boxed{4}$

$$\vec{p} = \lambda\,(\vec{u} \times \vec{v}) \times (\vec{w} \times \vec{x}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -6 & -2 & 1 \\ 1 & 1 & 2 \end{vmatrix} \lambda = \lambda\,(-5, 13, -4)$$ $\boxed{2}$

$|\vec{p}| = 1 \qquad 1 = \vec{p} \cdot \vec{p} = \lambda(25 + 169 + 16) \Rightarrow \quad \lambda = \pm \frac{1}{\sqrt{210}}$ $\boxed{2}$

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4) i) The normal form is $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad$ with $a = \frac{k}{1-e^2}$; $b = \frac{k}{\sqrt{1-e^2}}$

$\Rightarrow \quad a = \frac{9}{5} \frac{1}{1 - (\frac{4}{5})^2} = \frac{9}{5} \frac{1}{9/25} = 5 \quad ; \quad b = \frac{9}{5} \frac{1}{\sqrt{9/25}} = 3$

$\Rightarrow \quad \frac{x^2}{25} + \frac{y^2}{9} = 1$ $\boxed{2}$

ii) Line ; $\quad y = m x + c \qquad Q \in L \Rightarrow \quad 5 = c$

$P(x_0, y_0) \in L \Rightarrow \quad y_0 = m x_0 + c \Rightarrow m = \frac{y_0 -}{x_0}$

$\Rightarrow$ line through P and Q ; $\quad y = \frac{y_0 - 5}{x_0} x + 5 \quad \Leftrightarrow \quad \frac{y}{5} + \frac{5 - y_0}{5 x_0} x = 1$

Compare with the equation of the tangent through $P(x_0, y_0)$ on the ellipse $\quad \frac{x x_0}{25} + \frac{y y_0}{9} = 1$

$\Rightarrow \quad \frac{x_0}{25} = \frac{5 - y_0}{5 x_0} \quad \wedge \quad \frac{y_0}{9} = \frac{1}{5} \quad \Rightarrow \quad y_0 = \frac{9}{5} \quad \Rightarrow \frac{5 x_0^2}{25} = 5 - \frac{9}{5} \quad \Rightarrow x_0 =$

$\Rightarrow \quad \pm \frac{4x}{25} + \frac{y}{5} = 1 \quad \Rightarrow \quad \pm 4x + 5y - 25 = 0$ $\boxed{6}$

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5) i) $A(3,1,2)$ $\quad B(-1,2,0)$ $\quad \Rightarrow \vec{AB} = -4\vec{i} + \vec{j} - 2\vec{k}$

$\Rightarrow$ equation of the line through A and B

$$\mathcal{L}: \frac{x-3}{-4} = \frac{y-1}{1} = \frac{z-2}{-2} = \lambda$$

$\Rightarrow P(-4\lambda+3, \lambda+1, -2\lambda+2) \in \mathcal{L}$

$\Rightarrow P \in xy$-plane for $\lambda = 1$

$\Rightarrow$ The point of intersection is $\underline{P(-1,2,0)}$. $\boxed{4}$

ii) The line intersects the plane $\mathcal{P}: 2x - 4y - z = 30$ when

$$2(-4\lambda+3) - 4(\lambda+1) - (-2\lambda+2) = 30$$

$\Leftrightarrow \quad -8\lambda + 6 - 4\lambda - 4 + 2\lambda - 2 = 30 \quad \Rightarrow -10\lambda = 30 \Rightarrow \underline{\lambda = -3}$

$\Rightarrow$ The point of intersection: $P(-4(-3)+3, -3+1, -2(-3)+2) \in \mathcal{P}$

$\underline{P(15,-2,8) \in \mathcal{P}}$ $\boxed{4}$ $\quad$ 8

6) Normal vector to $\mathcal{P}_1$: $\vec{n}_1 = 3\vec{i} - \vec{j} + \vec{k}$

normal vector to $\mathcal{P}_2$: $\vec{n}_2 = -\vec{i} + 2\vec{j} + 6\vec{k}$

$\Rightarrow \vec{n}_1 \times \vec{n}_2$ is parallel to $\mathcal{L} = \mathcal{P}_1 \cap \mathcal{P}_2$

$$\Rightarrow \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 1 \\ -1 & 2 & 6 \end{vmatrix} = \vec{i}\begin{vmatrix} -1 & 1 \\ 2 & 6 \end{vmatrix} - \vec{j}\begin{vmatrix} 3 & 1 \\ -1 & 6 \end{vmatrix} + \vec{k}\begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix} = -8\vec{i} - 19\vec{j} + 5\vec{k}$$

$\boxed{4}$

Any point on $\mathcal{L}$ satisfies; i.e. take $z = 0$

$$\left.\begin{array}{l} 3x - y = 17 \\ -x + 2y = -14 \end{array}\right\} \Rightarrow 5x = 20 \Rightarrow \underline{x = 4} \Rightarrow \underline{y = -5}$$

$\Rightarrow$ The line of intersection is

$$\mathcal{L}: \frac{x-4}{-8} = \frac{y+5}{-19} = \frac{z}{5} = \lambda$$

$\boxed{4}$ $\quad$ 8

7/ i) Axiom 1 (line axiom):

Through any two distinct points P and Q there is

3 exactly one line $\mathcal{L} = \overleftrightarrow{PQ}$.

Axiom 2 (plane axiom):

Through any three noncollinear points there is

3 exactly one plane.

Axiom 3 (dimension axiom):

Any line contains at least two distinct points.

3 Any plane contains at least two distinct lines.

There are at least 2 distinct planes in space.

Axiom 4 (line-plane intersection axiom):

If two distinct points P_1, P_2 on a line $\mathcal{L}$ lie

3 in a plane $\mathcal{P}$, then the whole line lies in $\mathcal{P}$.

$(P_1, P_2 \in \mathcal{L}, \mathcal{P} \Rightarrow \mathcal{L} \in \mathcal{P})$

Axiom 5 (parallel axiom):

For a given point P and line $\mathcal{L}$ there is one and

3 only one line $\mathcal{L}'$ which passes through P and is

parallel to $\mathcal{L}$.

ii) Proof: (seen)

- from axiom 3 $\Rightarrow$ $\exists$ at least two distinct points $Q, R \in$
- 11 P, Q, R are not collinear since the unique line $\mathcal{L}$ through Q and R does not pass through P
- 26 from axiom 2 $\Rightarrow$ there is a unique plane $\mathcal{P}$ through $P, Q,$
- $\mathcal{P}$ contains two points of $\mathcal{L}$, namely Q, R $\Rightarrow$ from axiom

$\mathcal{L} \in \mathcal{P}$ q.e.

8) $A(3,-1,0)$ $\quad B(2,1,8)$ $\quad C(2,3,4)$

i) Take $P(x,y,z)$ to be an arbitrary point in the plane $\mathcal{P}$.

The following vectors are in the plane

$$\vec{AB} = -\vec{i} + 2\vec{j} + 8\vec{k} \in \mathcal{P}$$

$$\vec{AC} = -\vec{i} + 4\vec{j} + 4\vec{k} \in \mathcal{P}$$

$$\vec{AP} = (x-3)\vec{i} + (y+1)\vec{j} + z\vec{k} \in \mathcal{P}$$

The vector $\vec{AB} \times \vec{AC}$ is perpendicular to $\mathcal{P}$

$$\Rightarrow \quad \vec{AP} \cdot (\vec{AB} \times \vec{AC}) = 0$$

$$\Rightarrow \quad \begin{vmatrix} (x-3) & (y+1) & z \\ -1 & 2 & 8 \\ -1 & 4 & 4 \end{vmatrix} = 0$$

$$\Leftrightarrow \quad (x-3)(8-32) - (y+1)(-4+8) + z(-4+2) = 0$$

$$\Leftrightarrow \quad -24x + 72 - 4y - 4 - 2z = 0$$

$$\Leftrightarrow \quad \underline{\underline{\mathcal{P}: \quad 12x + 2y + z - 34 = 0}}$$

$\boxed{9}$

ii) The shortest distance for any point $P(x_0, y_0, z_0)$ from a plane $ax + by + cz = 0$ is

$$\delta = \left| \frac{a x_0 + b y_0 + c z_0 + d}{\sqrt{a^2 + b^2 + c^2}} \right|$$

$\boxed{2}$

$\Rightarrow$ The distance to the origin is therefore $\delta = \left| \frac{-34}{\sqrt{12^2 + 2^2 + 1}} \right| = \frac{34}{\sqrt{14}}$

iii) $P(2\lambda - 1, 5\lambda - 15, \lambda + 6) \in \mathcal{L}_1$

$\Rightarrow P \in \mathcal{P}$ when $12(2\lambda - 1) + 2(5\lambda - 15) + \lambda + 6 - 34 = 0$

$$\Leftrightarrow \quad 24\lambda - 12 + 10\lambda - 30 + \lambda + 6 - 34 = 0$$

$\boxed{6}$

$$\Leftrightarrow \quad 35\lambda - 70 = 0 \quad \Rightarrow \quad \underline{\lambda = 2}$$

$\Rightarrow$ The point of intersection is $P(3, -5, 8)$.

iv) $L_2: \dfrac{x-5}{4} = \dfrac{y+8}{-2} = \dfrac{z-11}{5}$

$\Rightarrow P_2(4\mu+5, -2\mu-8, 5\mu+11) \in L_2$

$P_1(2\lambda-1, 5\lambda-15, \lambda+6) \in L_1$

For the lines to intersect we require $P_1 = P_2$

$\therefore$ (1) $4\mu+5 = 2\lambda-1$

(2) $-2\mu-8 = 5\lambda-15$

(3) $5\mu+11 = \lambda+6$

(1) + 2(2): $-11 = 12\lambda - 31 \Rightarrow \lambda = \frac{20}{12} = \frac{5}{3}$

$\Rightarrow$ (1): $4\mu+5 = \frac{10}{3} - 1 \Rightarrow \mu = -\frac{2}{3}$

check (3): LHS: $5\cdot\left(-\frac{2}{3}\right) + 11 = \frac{23}{3}$

RHS: $\frac{5}{3} + 6 = \frac{23}{3}$

$\Rightarrow L_1$ and L_2 intersect

The point of intersection is $P_1 = P_2\left(2\cdot\frac{5}{3}-1, 5\cdot\frac{5}{3}-15, \frac{5}{3}+6\right)$

$= P\left(\frac{7}{3}, -\frac{20}{3}, \frac{23}{3}\right)$ 9

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9)

$S_1: x^2+y^2+z^2 = 9$

$S_2: x^2+(y+6)^2+z^2 = 36$

$\mathcal{P}: \lambda x + \mu y + z = 6$

i) - The distance d_1 of the centre C_1 of S_1, i.e. the origin, to the plane $\mathcal{P}$ is:

$$d_1 = \left|\frac{-6}{\sqrt{\lambda^2+\mu^2+1}}\right|$$

- For the point in $\mathcal{P}$ to be on the sphere as well we need

$d_1 = 3 \Rightarrow 3\sqrt{\lambda^2+\mu^2+1} = 6$

$\Rightarrow \lambda^2+\mu^2+1 = 2^2 \Rightarrow \lambda^2+\mu^2 = 3$ 8

ii) - The centre of S_2 is $C_2(0,-6,0)$

- The distance d_2 of C_2 to $\mathcal{P}$ is

$$d_2 = \left| \frac{-6\mu - 6}{\sqrt{\lambda^2+\mu^2+1}} \right|$$

- For this point to be on S_2 as well we require $d_2 = 6$

$$\Rightarrow \quad 6\sqrt{\lambda^2+\mu^2+1} = 6(\mu+1)$$

$$\Rightarrow \quad \lambda^2+\mu^2+1 = \mu^2+2\mu+1 \quad \Rightarrow \quad \underline{\underline{\lambda^2 = 2\mu}}$$ $\boxed{8}$

iii) For $\mathcal{P}$ to be a tangent plane to S_1 and S_2 we have to solve

$$\lambda^2+\mu^2 = 3 \quad \wedge \quad \lambda^2 = 2\mu$$

$$\Rightarrow \quad \mu^2 + 2\mu - 3 = 0$$

$$\Rightarrow \quad \mu_{1/2} = -1 \pm \sqrt{1+3} = -1 \pm 2$$

$\because \quad \lambda^2 = 2\mu > 0$ discard the minus sign

$$\Rightarrow \quad \underline{\mu = 1} \quad \Rightarrow \quad \underline{\lambda = \pm\sqrt{2}}$$

$\boxed{8}$

$\Rightarrow \quad \mathcal{P}: \quad \pm\sqrt{2}\,x + y + z = 6$ is the tangent plane to S_1, S_2

iv) For instance $A(0,0,6)$ and $B(0,6,0)$ are in $\mathcal{P}$.

$$\Rightarrow \quad \vec{AB} = (0,6,-6) \in \mathcal{P}$$

$\boxed{2}$

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