Solutions Complex Variable

All the questions cover standard material seen in the lectures and/or in the coursework. Only minor changes have been made compared to seen examples.

1. (a) (**seen**) Let f be analytic inside and on a closed simple contour C (oriented positively) that encloses the point z_0 then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz, \quad n = 0, 1, 2, \ldots$$

[**2**]

(b) (**unseen**) Each time, we use Cauchy's integral formula for an appropriate function and an appropriate contour.

i. The first task is to identify the singularities of the function we want to integrate. They are located at the solutions of the equation $z^3 + 1 = 0$. So we solve $z^3 = -1$ using the standard method of polar form. Setting $z = re^{i\theta}$ with $r > 0$ and $-1 = e^{i\pi}$ we obtain $r^3 = 1$ so $r = 1$ and $3\theta = \pi + 2n\pi$, $n \in \mathbb{Z}$. Keeping only the three distinct solutions corresponding for instance to $\theta = \pi/3$, $\theta = -\pi/3$ and $\theta = \pi$ we get

$$z_1 = e^{i\frac{\pi}{3}} \quad , \quad z_2 = e^{-i\frac{\pi}{3}} \quad , \quad z_3 = -1\,.$$

[**3**]

Only z_3 is in the given contour so we use Cauchy's integral formula for $z_0 = -1$, $f(z) = \frac{3-z}{z^2-z+1}$ and $n = 0$ to get

$$\oint_C \frac{1-2z}{1+z^3}\, dz = 2i\pi f(-1) = \frac{8i\pi}{3}\,.$$

[**3**]

ii. For this case, the only singularity is $z = \frac{\pi}{6}$ and it is inside the contour. Given the power of the denominator, we see that we can use Cauchy's integral formula for the second derivative of the function $f(z) = \sin z$ which is entire. [**2**]

So we get

$$\oint_C \frac{\sin z}{(z-\frac{\pi}{6})^3} dz = \frac{2i\pi}{2!} \frac{d^2}{dz^2}(\sin z)|_{z=\frac{\pi}{6}} = -i\frac{\pi}{2}\,.$$

[**4**]

Turn over . . .

iii. In this case, there are two singularities $z_1 = \frac{\pi}{4}$ and $z_2 = \frac{\pi}{2}$. But only z_1 lies inside the given contour. We use Cauchy's integral formula for $z_0 = \frac{\pi}{4}$, $f(z) = \frac{\sin z}{z-\frac{\pi}{2}}^2$ and $n = 0$ **[2]**

$$\oint_C \frac{\sin z}{(z-\frac{\pi}{4})(z-\frac{\pi}{2})^2} dz = 2i\pi \left(\frac{\sin z}{(z-\frac{\pi}{2})^2} \right)_{z=\frac{\pi}{4}} = \frac{16i\sqrt{2}}{\pi}.$$

[4]

Remark: the use of the residue theorem is also accepted although the very first question gives a clear indication that Cauchy's integral formula works well here.

2. (**unseen**)
Let $u(x, y) = x^2 - y^2 + e^y \cos x$ for all $x, y \in \mathbb{R}$.

(a) We compute

$$u_x(x, y) = 2x - e^y \sin x \quad , \quad u_{xx}(x, y) = 2 - e^y \cos x \,,$$

and

$$u_y(x, y) = -2y + e^y \cos x \quad , \quad u_{yy}(x, y) = -2 + e^y \cos x \,.$$

So $u_{xx} + u_{yy} = 0$ for all $x, y \in \mathbb{R}$. [4]

(b) Define the function v by $v_y = u_x$ and $v_x = -u_y$ for all $x, y \in \mathbb{R}$ and $v(0, 0) = 0$. Find v. We solve the given partial differential equations for v. They read

$$\begin{cases} v_y = 2x - e^y \sin x \,, \\ v_x = 2y - e^y \cos x \,. \end{cases}$$

[2]

The first equation yields

$$v(x, y) = 2xy - e^y \sin x + g(x) \,,$$

where g is some function of x only. We determine it by inserting in the second equation which reduces to

$$g'(x) = 0 \,.$$

So g is a constant, say C, and

$$v(x, y) = 2xy - e^y \sin x + C \,.$$

Turn over ...

Now using $v(0,0) = 0$ yields $C = 0$. So finally,

$$v(x,y) = 2xy - e^y \sin x\,.$$

[8]

(c) The function whose real and imaginary parts are u and v reads

$$f(x,y) = x^2 - y^2 + e^y \cos x + i(2xy - e^y \sin x) = (x+iy)^2 + e^{y-ix}\,.$$

By construction of v, we see that u and v satisfy the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ for all $(x,y) \in \mathbb{R}^2$. And the partial derivatives u_x, u_y, v_x, v_y are continuous for all $(x,y) \in \mathbb{R}^2$. So f is analytic everywhere. It is an entire function. In terms of $z = x + iy$, we get

$$f(z) = z^2 + e^{-iz}\,.$$

Written in this form, it is now easy to see that f is indeed an entire function.

[6]

3. (**unseen**)

(a) We use the change of variables discussed in the lectures $z = e^{i\theta}$. Then $\cos\theta = \frac{1}{2}(z + \frac{1}{z})$, $\sin\theta = \frac{1}{2i}(z - \frac{1}{z})$ and $d\theta = \frac{dz}{iz}$. Inserting, The integral becomes the following contour integral in z along the unit circle C_0

$$I = \oint_{C_0} \frac{2}{(-1-i)z^2 + 2iz - 1 + i} dz\,,$$

which can be computed using the residue theorem. [**3**]

The poles of the function are

$$z_\pm = \frac{i \pm i\sqrt{3}}{1+i}\,.$$

[**3**]

z_- has modulus strictly less than 1 and z_+ has modulus strictly greater than 1 (compute them directly!). So only the residue at $z = z_-$ contributes and we get

$$I = 2i\pi \frac{2}{(-1-i)(z_- - z_+)}\,.$$

Now $z_- - z_+ = -\frac{2i\sqrt{3}}{1+i}$ so

$$I = \frac{2\pi}{\sqrt{3}}\,.$$

[**4**]

(b) The standard method for this type of integral is to defined $f(z) = \frac{1}{i+z^3}$ and to integrate it on the closed contour made of the line segment $[-R, R]$ and the semi-circle $C_R = \{z \in \mathbb{C}, z = R\, e^{i\theta}, 0 \leq \theta \leq \pi\}$ in the limit $R \to \infty$. Since $zf(z)$ tends to 0 as z tends to ∞, we know that the integral along C_R tends to 0. [**1**]

Now the poles of f that are inside the contour are all the poles in the upper-half plane. The poles of f are obtained by solving $i + z^3 = 0$. Using the usual method involving polar forms $z = \rho e^{i\theta}$ and $-i = e^{-i\frac{\pi}{2}}$. We obtain the general solution

$$z = e^{i\frac{(4q-1)\pi}{6}} \quad , \quad q \in \mathbb{Z}\,.$$

So the 3 distinct poles of f are $z_q = e^{i\frac{(4q-1)\pi}{6}}$ with $q = 0, 1, 2$. [**3**]

Among these, only the one with $q = 1$ is inside the contour. Finally, all the poles are simple for we can use the simple formula seen in the lectures for rational fractions. It takes the form $Res(f, z_q) = \left.\frac{1}{\frac{d}{dz}(i+z^3)}\right|_{z=z_q} = \frac{1}{3z_q^2}$ here. So using the residue theorem we get, putting everything together

$$J = 2i\pi Res(f, z_1) = \frac{2i\pi}{3e^{i\pi}} = -\frac{2i\pi}{3}\,.$$

[**6**]

4. (**unseen**) In this question, we compute

$$F(k) = \int_{-\infty}^{\infty} f(x)\, e^{ikx} \quad , \quad k \in \mathbb{R}$$

where $f(x) = \frac{x}{4+x^2}$.

(a) The idea is to integrate the function $f(z) = \frac{z}{4+z^2}$ around the usual semi-circle contour of radius R and then send R to infinity. The relevant lemma here ensuring that the contribution to the integral on the semi-circle vanishes is Jordan's lemma. [**1**]

The actual choice of the semi-circle, either in the upper or lower half-plane depends on the sign of k. First, if $k = 0$ then $F(0) = 0$ since the integrand is an odd function. [**2**]

If $k > 0$, we choose the same contour as in the previous question. If $k < 0$, we choose the contour made of the line segment $[-R, R]$ and the semi-circle $C_R = \{z \in \mathbb{C}, z = R\, e^{i\theta}, -2\pi \leq \theta \leq -\pi\}$ (careful with the orientation!). [**1**]

The poles of f are given by $4+z^2=0$. We obtain two simple poles $z_0 = 2i$ and $z_1 = -2i$. When $k>0$, only z_0 contributes. When $k<0$, only z_1 contributes. [**2**]

(b) Using the residue theorem in view of all the information in the first part, we get, for $k>0$

$$F(k) = 2i\pi Res(f,z_0) = 2i\pi\frac{e^{ikz_0}}{2} = i\pi e^{-2k}\,.$$

For $k<0$,

$$F(k) = -2i\pi Res(f,z_1) = -2i\pi\frac{e^{ikz_1}}{2} = -i\pi e^{2k}\,.$$

Summarizing,

$$F(k) = \begin{cases} i\pi e^{-2k} \quad , \quad k>0\,, \\ 0 \quad , \quad k=0\,, \\ -i\pi e^{2k} \quad , \quad k<0\,. \end{cases}$$

In particular, $F(-k) = -F(k)$, which is consistent in view of the parity of f.

[**12**]

(c) The value of

$$\int_{-\infty}^{\infty} f(x)\,\cos(kx)\,dx$$

coincides with the real part of $F(k)$ so is zero. [**2**]
In fact, this could have been seen directly from the parity of f and that of cos.

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