MA2603

CITY UNIVERSITY

London

BSc Honours Degree in Mathematical Science
Mathematical Science with Statistics
Mathematical Science with Computer Science
Mathematical Science with Finance and Economics
Mathematics and Finance

Part 2

Mathematical Methods

2011

Time allowed: 3 hours

Full marks may be obtained for correct answers to FIVE out of the SEVEN questions.
All necessary working must be shown.

1. Let f be the 2π periodic function defined on $[-\pi, \pi)$ by

$$f(x) = \sin\frac{x}{4}\,.$$

   (a) Find the Fourier coefficients of f.

[6]

   (b) Show that for all $x \in (-\pi, \pi)$,

$$\sin\frac{x}{4} = \sum_{n=1}^{\infty} \frac{(-1)^n 16n\sqrt{2}}{\pi(1-16n^2)}\sin nx\,.$$

[2]

   (c) State Parseval's theorem and use it to prove

$$\frac{\pi^2}{512}\left(1-\frac{2}{\pi}\right) = \sum_{n=1}^{\infty}\frac{n^2}{(1-16n^2)^2}\,.$$

[12]

2. Consider the surface of that part of the cone $4y^2 = x^2 + z^2$ which lies inside the cylinder $x^2 + z^2 = x$.

   (a) Giving reasons, choose which of the following parametrizations is appropriate for this surface

$$\begin{array}{rlcl} \overrightarrow{F}_{\pm}: & [0,1]\times[-\frac{\pi}{2},\frac{\pi}{2}] & \to & \mathbb{R}^3 \\ & (\rho,\theta) & \mapsto & \rho\cos\theta\,\vec{i} + \rho\sin\theta\,\vec{j} \pm \frac{\rho}{2}\,\vec{k} \end{array}$$

$$\begin{array}{rlcl} \overrightarrow{G}_{\pm}: & [0,\cos\theta]\times[-\frac{\pi}{2},\frac{\pi}{2}] & \to & \mathbb{R}^3 \\ & (\rho,\theta) & \mapsto & 2\cos\theta\,\vec{i} + 2\sin\theta\,\vec{j} \pm \frac{\rho}{2}\,\vec{k} \end{array}$$

$$\begin{array}{rlcl} \overrightarrow{H}_{\pm}: & [0,\cos\theta]\times[-\frac{\pi}{2},\frac{\pi}{2}] & \to & \mathbb{R}^3 \\ & (\rho,\theta) & \mapsto & \rho\cos\theta\,\vec{i} \pm \frac{\rho}{2}\,\vec{j} + \rho\sin\theta\,\vec{k} \end{array}$$

   where the subscript $\pm$ refers to the two symmetric parts of the surface.

[6]

(b) Using the previous parametrization, show that the surface area is

$$S = \frac{\pi\sqrt{5}}{4} .$$

[**14**]

3. (a) Using cylindrical coordinates, show that a parametrization of the closed curve C which is the intersection of the paraboloid $-2z = x^2 + y^2$ with the plane $y = z$ is

$$\vec{r}(\theta) = -\sin 2\theta\,\vec{\imath} - 2\sin^2\theta\,(\vec{\jmath} + \vec{k}) \quad , \quad \theta \in [0, \pi] .$$

(Discuss carefully the range for θ.)

[**7**]

(b) Let $\vec{V}(x, y, z) = x\,\vec{\imath} + (1 + y)\,\vec{\jmath}$ be a vector field. Compute the line integral

$$I = \int_C \vec{V} \cdot d\vec{r} .$$

[**6**]

(c) Show that $\vec{V} = \vec{\nabla}\phi$ for a scalar field ϕ that you should find. Hence, deduce the value of I and check it against your previous result.

[**7**]

4. Let $\vec{V}(x, y, z) = x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}$ be a vector field. Let S be the surface of the cylinder $(x-2)^2 + \frac{(y-1)^2}{3} = 1$ between the $z = 0$ plane and the plane $y + 4z = 6$.

(a) Compute $\vec{\nabla} \cdot \vec{V}$.

[**2**]

(b) Giving reasons, choose which of the following parametrizations describes the volume inside the cylinder and the two planes.

$$\begin{aligned} \vec{F}: \quad [0,1] \times [0, 2\pi] \times [0, z_0(\rho, \theta)] \quad &\rightarrow \quad \mathbb{R}^3 \\ (\rho, \theta, z) \quad &\mapsto \quad (2 + \rho\cos\theta)\,\vec{\imath} + (1 + \sqrt{3}\rho\sin\theta)\,\vec{\jmath} + z\,\vec{k} \end{aligned}$$

where $z_0(\rho, \theta) = \frac{1}{4}(5 - \sqrt{3}\rho\sin\theta)$.

$$\begin{aligned} \vec{G}: \quad [0,1] \times [0, 2\pi] \times [0, \rho^2] \quad &\rightarrow \quad \mathbb{R}^3 \\ (\rho, \theta, z) \quad &\mapsto \quad (2 + \rho\cos\theta)\,\vec{\imath} + (1 + \sqrt{3}\rho\sin\theta)\,\vec{\jmath} + z\,\vec{k} \end{aligned}$$

$$\begin{array}{rccl}\vec{H}: & [0,\sqrt{3}]\times[0,2\pi] & \to & \mathbb{R}^3 \\ & (\rho,\theta) & \mapsto & \rho\cos\theta\,\vec{i}+\rho\sin\theta\,\vec{j}+\rho^2\vec{k}\end{array}$$

[4]

(c) Show that the surface integral

$$I=\iint_S \vec{V}\cdot d\vec{S}\,,$$

equals $\frac{15\pi\sqrt{3}}{4}$.

[14]

5. Define $\gamma=\alpha+2i$ and let

$$f(z)=(\gamma z+\bar{\gamma}\bar{z})^2+2i(\bar{\gamma}z+\gamma\bar{z})\,,$$

where $\bar{z}$ (respectively $\bar{\gamma}$) is the complex conjugate of z (respectively γ).

(a) Find the real and imaginary parts of f in terms of α, x and y where $z=x+iy$.

[6]

(b) For which values of α is f differentiable? For these values, find where it is differentiable.

[12]

(c) Is f analytic anywhere?

[2]

6. Let f be defined by

$$f(z)=\frac{1}{3z^2+(1-6i)z-2i}\,.$$

(a) Find the poles z_1 and z_2 of f such that $|z_1|<|z_2|$. (Simplify your answer as much as possible to obtain the poles in the form $a+ib$ where a and b are rational numbers only).

[2]

(b) Give the series expansion of f in the following three regions, stating in each case if it is a Taylor or Laurent series.

i. $|z| < |z_1|$.

[7]

ii. $|z_1| < |z| < |z_2|$.

[5]

iii. $|z - z_1| < |z_2 - z_1|$.

[6]

7. (a) Explaining your method, compute

$$J_1 = \int_{-\infty}^{\infty} \frac{dx}{1+x^4} \,.$$

and

$$J_2 = \int_{-\infty}^{\infty} \frac{x^2 dx}{1+x^4} \,.$$

Simplify your answer until you get an expression involving real numbers only.

[16]

(b) Deduce the value of

$$J = \int_{-\infty}^{\infty} \frac{x^3 + x^2 + x + 1}{1+x^4} \, dx \,.$$

[4]

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