City University
London

MA2600: Complex Variable

Time allowed: 2 hours

Full marks may be obtained for correct answers to THREE of the FOUR questions.
If more than THREE questions are answered, the best THREE marks will be credited.

Turn over ...

1. (a) State Cauchy's integral formula together with the conditions required for it to hold.

   (b) Evaluate the following integrals

   i.
   $$\oint_C \frac{1-2z}{1+z^3}\,dz$$
   where C is the rectangle with opposite corners at $\pm(2+\frac{i}{4})$.

   ii.
   $$\oint_C \frac{\sin z}{(z-\frac{\pi}{4})^3}dz$$
   where C is the unit circle.

   iii.
   $$\oint_C \frac{\sin z}{(z-\frac{\pi}{4})^2(z-\frac{\pi}{2})}dz$$
   where C is the unit circle.

2. Let f be defined by
$$f(z) = \frac{2}{z^2-(2i+4)z+8i}\,.$$

   (a) Find the poles of f. Call them z_1 and z_2 such that $|z_1| < |z_2|$.

   (b) Give the series expansion of f in the following three regions, stating in each case if it is a Taylor or Laurent series.

   i. $|z| < |z_1|$.

   ii. $|z_1| < |z| < |z_2|$.

   iii. $|z - z_1| < |z_2 - z_1|$.

3. Explaining your method, compute

   (a)
   $$I = \int_0^{2\pi} \frac{d\theta}{\cos\theta - \sin\theta - i}\,.$$
   Simplify your answer until you get a purely imaginary number.

Turn over ...

(b)

$$J = \int_{-\infty}^{\infty} \frac{dx}{i + x^5} \,.$$

Simplify your answer until you get a purely imaginary number.

4. In this question, we compute

$$F(k) = \int_{-\infty}^{\infty} f(x)\, e^{ikx} \quad , \quad k \in \mathbb{R} \,,$$

where $f(x) = \frac{x}{1+x^4}$.

(a) Describe the contour that should be used to compute this integral (paying attention to the various cases) and state the relevant lemma. Find the poles of f and discuss their contributions in the various cases.

(b) Find $F(k)$.

(c) Deduce the value of

$$\int_{-\infty}^{\infty} f(x)\, \cos(kx)\, dx \quad , \quad k \in \mathbb{R} \,.$$

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