City University
London

MA2604: Calculus and Vector Calculus

Time allowed: 3 hours

Full marks may be obtained for correct answers to FIVE of the EIGHT questions.
If more than FIVE questions are answered, the best FIVE marks will be credited.
At least TWO questions should be attempted in each part.

Each question carries 20 marks

Part A: Calculus

1. (a) Show that if

$$f(x,y) = \frac{2x(y-1)^2}{x^3+(y-1)^3}, \quad (x,y) \neq (0,1)$$

then the limit of $f(x,y)$ as $(x,y) \to (0,1)$ does not exist. [**8**]

(b) If $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are the partial derivatives of any function $f(x,y)$ and $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ are the partial derivatives of f viewed as a function of two new variables s and t, show that if

$$s = e^{x^2+y^2}, \quad t = e^{xy}$$

then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2\left(s \ln s \frac{\partial f}{\partial s} + t \ln t \frac{\partial f}{\partial t}\right).$$

By evaluating each side directly, verify this result in the case where $f = (x+y)^2$. [**12**]

2. (a) Find the stationary points of the function

$$f(x,y) = 3x^4 - 32x^3 + 24xy^2 - 24y^2 - 54x^2 + 1$$

[**7**]

(b) Determine whether they are maxima, minima or saddle points. [**8**]

(c) Give the Taylor series expansion of the function f about the point $x=-1, y=0$ up to and including second order terms. [**5**]

3. (a) Sketch the region of integration in the x, y plane of the integral

$$I = \int_{x=0}^{x=1} dx \int_{y=x^2}^{y=1} x \cos\left(\frac{\pi y^2}{2}\right) dy.$$

By changing the order of integration, evaluate I. [**10**]

Turn over . . .

(b) By transforming to new coordinates $u = xy, v = xy^3$, or otherwise, evaluate the integral

$$\int\int_R xy(1 + xy^3)dxdy,$$

where R is the region in the first quadrant of the x, y plane bounded by the curves $xy = 1, xy = 4, xy^3 = 1$ and $xy^3 = 4$. [**10**]

4. You may assume that the general solution of the differential equation

$$\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + b(x)y = r(x),$$

where a, b and r are functions of x, can be written in the form

$$y = \left(A - \int^x \frac{ry_2}{W}dx\right) y_1 + \left(B + \int^x \frac{ry_1}{W}dx\right) y_2,$$

where A and B are arbitrary constants and $W \neq 0$ is the Wronskian of the complementary solutions y_1 and y_2 of the homogeneous version of the equation. Use this result to find the general solution of the equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = e^{2x}\text{cosec}x.$$

[**14**]

Hence find the solution for y in the region $\frac{\pi}{4} \leq x \leq \frac{\pi}{2}$ that satisfies the boundary conditions $y(\frac{\pi}{4}) = 0$ and $y(\frac{\pi}{2}) = 0$. [**6**]

Part B: Vector Calculus

5. (a) Give a parametrization of the portion of the sphere $x^2 + y^2 + z^2 = 4^2$ between the two planes $z = 3$ and $z = -3$. [**10**]

   (b) Use your parametrization to show the corresponding surface area is 48π. [**10**]

6. (a) Give a parametrization of the (finite) volume that lies inside the following two paraboloids: $y = x^2 + z^2 + 2$ and $y = 2(x^2 + z^2) - 1$. [**10**]

   (b) Use your parametrization to find the (finite) volume V that lies inside the two paraboloids. [**10**]

7. Consider the ellipsoid $\mathcal{E}$ with Cartesian equation in standard form

$$\frac{x^2}{4} + \frac{y^2}{10} + \frac{z^2}{5} = 1\,.$$

   (a) Find all the points on $\mathcal{E}$ where the tangent plane is orthogonal to the plane of equation $y = 0$. The solution is a curve C whose Cartesian equations you should give. [**10**]

   (b) Find the line integral of the following vector field along C:

$$\vec{V}(x, y, z) = -z\vec{i} + x\vec{k}\,.$$

[**10**]

8. Let $\vec{V}(x, y, z) = -z\,\vec{i} + y/3\,\vec{j} + 2y\,\vec{k}$ be a vector field. Let S be the surface of the cylinder $\frac{(x+1)^2}{9} + (y-3)^2 = 1$ between the $z = 0$ plane and the plane $2x + 3y - 2z = -1$.

   (a) Compute $\vec{\nabla} \cdot \vec{V}$. [**2**]

   (b) Giving reasons, choose which of the following parametrizations describes the volume inside the cylinder and the two planes.

$$\begin{array}{llll} \vec{r}_1 : & [0,1] \times [0, 2\pi] \times [0, \rho^2] & \to & \mathbb{R}^3 \\ & (\rho, \theta, z) & \mapsto & (3\rho\cos\theta - 1)\,\vec{i} + (3 + \rho\sin\theta)\,\vec{j} + z\,\vec{k} \end{array}$$

Turn over ...

$$\begin{aligned} \vec{r}_2 : \quad [0, ab] \times [0, 2\pi] &\rightarrow \mathbb{R}^3 \\ (\rho, \theta) &\mapsto \rho \cos\theta \, \vec{i} + \rho \sin\theta \, \vec{j} + \rho^2 \vec{k} \end{aligned}$$

$$\begin{aligned} \vec{r}_3 : \quad [0, 1] \times [0, 2\pi] \times [0, z_0(\rho, \theta)] &\rightarrow \mathbb{R}^3 \\ (\rho, \theta, z) &\mapsto (3\rho\cos\theta - 1)\, \vec{i} + (3 + \rho\sin\theta)\, \vec{j} + z\, \vec{k} \end{aligned}$$

where $z_0(\rho, \theta) = (8 + 6\rho\cos\theta + 3\rho\sin\theta))/2$. **[6]**

(c) Show that the surface integral

$$I = \iint_S \vec{V} \cdot d\vec{S} \,,$$

equals 4π. **[12]**

Internal Examiner: Prof P. Daniels, Dr V. Caudrelier
External Examiner: Prof J. Rickard, Prof J. Lamb