Calculus and vector calculus set task

Please notice that you must attempt all questions.

1. (a) Find and classify the stationary points (maxima, minima and saddle points) of the function
$$f(x,y) = 2x^3 - 6xy + y^2.$$
(b) Compute the Taylor's expansion of the function
$$f(x,y) = f(x,y) = xe^{-x^2+y^2}.$$
around the point $(1/\sqrt{2}, 0)$ including up to second order terms. What can you conclude from the form of the expansion about the nature of the point $(1/\sqrt{2}, 0)$?

2. Determine the functions $u_1(x)$, $u_2(x)$ such that $y(x) = c_1u_1(x) + c_2u_2(x)$ is the general solution of the following homogeneous second-order differential equation
$$y'' + y = 0,$$
where c_1, c_2 are arbitrary constants. Show that the Wronskian of u_1, u_2 is nowhere zero.

   Use the method of variation of parameters to find a particular solution of the inhomogeneous second-order differential equation
$$y'' + y = 3e^{-x} + x.$$
   Hence determine the general solution of this inhomogeneous equation.

3. Consider the ellipsoid E of equation
$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{12} = 1\,,$$
and the plane P of equation $y + z = 0$. Let Γ be the intersection of E and P.

   (a) Find a parametrization for Γ.

   (b) Let $\vec{V}(x,y,z) = -y\,\vec{i} + x\,\vec{j} - x\,\vec{k}$. Compute the line integral of $\vec{V}$ along Γ.

4. Let C be a curve in two dimensions parametrized by
$$\begin{cases} x(t) = \cos(f(t))\,, \\ y(t) = \sin(f(t))\,, \end{cases} \equiv \vec{r}(t)$$
where f is a smooth function of $t \in [a,b]$. Define the vector field $\vec{V}$ by
$$\vec{V}(x,y) = \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}\,,$$
and the following line integral along C by
$$I = \frac{1}{2\pi}\int_C \vec{V} \cdot d\vec{r}.$$

   Please turn over

(a) Show that

$$I = \frac{1}{2\pi}(f(b) - f(a))\,.$$

Deduce that if C is a closed curve then I is an integer.

(b) Set $f(t) = 2t + 3$ and $a = 0$, $b = 2\pi$. Check that the corresponding curve C is closed and compute I . What is this curve?