Solutions Calculus and Vector Calculus

1. (**Unseen**)

(a) Given the function $f(x,y) = x^2 \cos y$, with $x = x(t)$ and $y = y(t)$, we find (we use the notation $\dot{x} \equiv \frac{dx}{dt}$, $\dot{y} \equiv \frac{dy}{dt}$)

(i)

$$\frac{df}{dt} = 2x\dot{x}\cos y - x^2\dot{y}\sin y.$$

[4]

(ii)

$$\frac{d^2 f}{dt^2} = (2\dot{x}^2 + 2x\ddot{x} - x^2\dot{y}^2)\cos y - (4x\dot{x}\dot{y} + x^2\ddot{y})\sin y.$$

[6]

(b) (i) The Jacobian matrix of the coordinate transformation is

$$J = \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix} = \begin{pmatrix} \cos t & -s\sin t \\ \cos(s-t) & -\cos(s-t) \end{pmatrix}.$$

[4]

(ii) The partial derivatives $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ are given by

$$\begin{aligned} \frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = (2x-y)\cos t - x\cos(s-t) = \\ &= (2s\cos t - \sin(s-t))\cos t - s\cos t\cos(s-t) \\ \frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = -(2x-y)s\sin t + x\cos(s-t) = \\ &= -(2s\cos t - \sin(s-t))s\sin t + s\cos t\cos(s-t). \end{aligned}$$

[6]

2. (**Unseen**)

(i) The stationary points of the function $f(x,y) = xy - \log(x^2+y^2)$ are given by the solutions of the equations

$$\begin{aligned} \frac{\partial f}{\partial x} &= y - \frac{2x}{x^2+y^2} = 0, \\ \frac{\partial f}{\partial y} &= x - \frac{2y}{x^2+y^2} = 0. \end{aligned}$$

Turn over ...

[2]

Note that the point $(0,0)$ is not in the domain of definition of the function. Solving the system, we find

$$\frac{x}{y} = \frac{y}{x} \Longrightarrow x^2 = y^2 \Longrightarrow y = \pm x,$$

and thus

$$x = \pm\frac{2x}{2x^2} \Longrightarrow x^2 = \pm 1.$$

This equation has no real solutions if we choose the $-$ sign. Thus we must choose the $+$ sign and we finally find $x = y = \pm 1$. Thus f has two stationary points:

$$P_1 = (1,1), \qquad P_2 = (-1,-1).$$

[4]

(ii) In order to determine the nature of the stationary points, we need to calculate the second derivatives:

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= -\frac{2}{x^2+y^2} + \frac{4x^2}{(x^2+y^2)^2} = \frac{2(x^2-y^2)}{(x^2+y^2)^2},\\
\frac{\partial^2 f}{\partial y^2} &= -\frac{2}{x^2+y^2} + \frac{4y^2}{(x^2+y^2)^2} = \frac{-2(x^2-y^2)}{(x^2+y^2)^2},\\
\frac{\partial^2 f}{\partial x \partial y} &= 1 + \frac{4xy}{(x^2+y^2)^2}.
\end{aligned}$$

[4]

At the point $P_1 = (1,1)$ we find

$$f_{xx} = f_{yy} = 0, \quad f_{xy} = 2 \quad \Longrightarrow \quad f_{xx}f_{yy} - f_{xy}^2 = -2 < 0.$$

We conclude that P_1 is a saddle point. [2]

Similarly, at the point $P_2 = (-1,-1)$ we find

$$f_{xx} = f_{yy} = 0, \quad f_{xy} = 2 \quad \Longrightarrow \quad f_{xx}f_{yy} - f_{xy}^2 = -2 < 0.$$

We conclude that also P_2 is a saddle point. [2]

(iii) The Taylor expansion of a function f around a point $P = (x_0, y_0)$ up to and including second order terms is given by

$$\begin{aligned}
f(x,y) = {} & f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + \\
& \frac{1}{2}\left[f_{xx}(x_0,y_0)(x-x_0)^2 + f_{yy}(x_0,y_0)(y-y_0)^2 + 2f_{xy}(x_0,y_0)(x-x_0)(y-y_0)\right] + \ldots
\end{aligned}$$

Turn over ...

Thus the expansion around $P_1 = (1, 1)$ is given by

$$f(x, y) = 1 - \log 2 + 2(x - 1)(y - 1),$$

[3]

and the expansion around $P_2 = (-1, -1)$ is given by

$$f(x, y) = 1 - \log 2 + 2(x + 1)(y + 1).$$

[3]

3. (**Unseen**)

(i) We calculate

$$\begin{aligned} y &= \frac{\cos x}{\sqrt{x}}, \\ y' &= -\frac{\sin x}{\sqrt{x}} - \frac{\cos x}{2x^{3/2}}, \\ y'' &= -\frac{\cos x}{\sqrt{x}} + \frac{\sin x}{x^{3/2}} + \frac{3\cos x}{4x^{5/2}}. \end{aligned}$$

[2]

Plugging into the left-hand side of the differential equation, we find

$$-\frac{\cos x}{\sqrt{x}} + \frac{\sin x}{x^{3/2}} + \frac{3\cos x}{4x^{5/2}} - \frac{1}{x}\left[\frac{\sin x}{\sqrt{x}} + \frac{\cos x}{2x^{3/2}}\right] + \left(1 - \frac{1}{4x^2}\right)\frac{\cos x}{\sqrt{x}} = 0.$$

[2]

(ii) If $y_1(x)$ is a solution of the second order homogeneous ODE

$$y'' + a(x)y' + b(x)y = 0,$$

then a second solution is given by

$$y_2(x) = y_1(x)Y(x)$$

where

$$Y(x) = \int^x \frac{e^{-\int^t a(s)ds}}{y_1^2(t)} dt.$$

[2]

In our case we find

$$Y(x) = \int^x \frac{t}{\cos^2 t} e^{-\int^t \frac{1}{s} ds} = \int^x \frac{t}{\cos^2 t} e^{-\log t} = \int^x \frac{1}{\cos^2 t} = \tan x,$$

and thus

$$y_2(x) = y_1(x)Y(x) = y_1(x)\tan x = \frac{\sin x}{\sqrt{x}}.$$

[2]

(iii) The Wronskian is given by

$$W(x) = \det\begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix} = \det\begin{pmatrix} \frac{\cos x}{\sqrt{x}} & \frac{\sin x}{\sqrt{x}} \\ -\frac{\sin x}{\sqrt{x}} - \frac{\cos x}{2x^{3/2}} & \frac{\cos x}{\sqrt{x}} - \frac{\sin x}{2x^{3/2}} \end{pmatrix} = \frac{1}{x},$$

which is always different from 0. [4]

(iv) According to the method of variation of parameters, a particular solution is obtained as

$$y_p(x) = U(x)y_1(x) + V(x)y_2(x),$$

where [2]

$$U(x) = -\int^x \frac{y_2(t)r(t)}{W(t)}dt = -\int^x \frac{(\sin t/\sqrt{t})t^{-1/2}}{1/t}dt = -\int^x \sin t\, dt = \cos x,$$
$$V(x) = \int^x \frac{y_1(t)r(t)}{W(t)}dt = \int^x \frac{(\cos t/\sqrt{t})t^{-1/2}}{1/t}dt = \int^x \cos t\, dt = \sin x.$$

Thus we get

$$y_p(x) = \cos x\frac{\cos x}{\sqrt{x}} + \sin x\frac{\sin x}{\sqrt{x}} = \frac{1}{\sqrt{x}}$$

[4]

(v) The general solution is

$$y(x) = A\frac{\cos x}{\sqrt{x}} + B\frac{\sin x}{\sqrt{x}} + \frac{1}{\sqrt{x}},$$

where A and B are arbitrary real numbers. [2]

4. (**Unseen** but the same question with the variable x and y swapped was in the coursework.)

   (a) The interior of the torus is described by the inequality $y^2 + (3 - \sqrt{x^2+z^2})^2 \leq 4$. We use the usual substitution $y = \rho\sin u$ and $3 - \sqrt{x^2+z^2} = \rho\cos u$ with $\rho \geq 0$ and the condition becomes $\rho^2 \leq 4$ with no constraint on u. Therefore $\rho \in [0, 2]$ and, since cos and sin are

2π-periodic, to get the full torus only once, we can take $u \in [0, 2\pi]$. Now looking at $3 - \sqrt{x^2 + z^2} = \rho \cos u$, we rearrange it into $x^2 + z^2 = (3 - \rho \cos u)^2$. Using again the standard identity $\cos^2 v + \sin^2 v = 1$ suggests setting $x = \cos v(3 - \rho \cos u)$ and $z = \sin v(3 - \rho \cos u)$. The equality is then satisfied with no constraint on v so, as before, we take $v \in [0, 2\pi]$. Summarizing, we get the parametrization

$$\vec{r}(\rho, u, v) = \begin{pmatrix} \cos v(3 - \rho \cos u) \\ \rho \sin u \\ \sin v(3 - \rho \cos u) \end{pmatrix}, \quad (\rho, u, v) \in [0, 2] \times [0, 2\pi] \times [0, 2\pi] .$$

[10]

(b) Using our parametrization, we compute

$$\begin{aligned} \left(\frac{\partial \vec{r}}{\partial \rho} \times \frac{\partial \vec{r}}{\partial u} \right) \cdot \frac{\partial \vec{r}}{\partial v} &= \begin{vmatrix} -\cos v \cos u & \rho \cos v \sin u & -\sin v(3 - \rho \cos u) \\ \sin u & \rho \cos u & 0 \\ -\sin v \cos u & \rho \sin v \sin u & \cos v(3 - \rho \cos u) \end{vmatrix} \\ &= -\rho(3 - \rho \cos u) . \end{aligned}$$

So the volume element is $dV = \rho |3 - \rho \cos u| \, d\rho \, du \, dv = \rho(3 - \rho \cos u) \, d\rho \, du \, dv$ since $\rho \cos u \leq 2$ as can be seen from the ranges on the parameters. **[4]**

Then,

$$\begin{aligned} V &= \int_0^2 \int_0^{2\pi} \int_0^{2\pi} \rho(3 - \rho \cos u) \, d\rho \, du \, dv \\ &= 12\pi^2 \int_0^2 \rho \, d\rho - 2\pi \int_0^2 \int_0^{2\pi} \rho^2 \cos u \, d\rho \, du \\ &= 24\pi^2 . \end{aligned}$$

[6]

5. (**Unseen** but the same question with the roles of y and z swapped was in last year's exam.)

Consider the ellipsoid $\mathcal{E}$ with Cartesian equation in standard form

$$\frac{x^2}{4} + \frac{y^2}{12} + \frac{z^2}{5} = 1 .$$

(a) The tangent plane to $\mathcal{E}$ at a point $P = (x, y, z)$ has normal vector given by $\vec{\nabla} f(x, y, z)$ where $f(x, y, z) = \frac{x^2}{4} + \frac{y^2}{12} + \frac{z^2}{5} - 1$. Here

$$\vec{\nabla} f(x, y, z) = \begin{pmatrix} x/2 \\ y/6 \\ 2z/5 \end{pmatrix} .$$

[**3**]

This vector should be orthogonal to the normal vector to the plane $z = 0$, that is the vector

$$\vec{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} .$$

[**2**]

So $\vec{n} \cdot \vec{\nabla} f(x, y, z) = 0$, which gives $z = 0$. [**1**]

So the points solutions of the required conditions satisfy

$$\frac{x^2}{4} + \frac{y^2}{12} = 1 \quad , \quad z = 0 \, .$$

This is the ellipse in the xOy plane corresponding to the intersection of the ellipsoid with the $z = 0$ plane. [**4**]

(b) We first find a parametrization for C. Setting $x = 2\cos\phi$ and $y = 2\sqrt{3}\sin\phi$ the equation of the ellipse is satisfied with no constraint on ϕ. To get the full ellipse once, it is enough to take $\phi \in [0, 2\pi]$. Therefore

$$\vec{r}(\phi) = \begin{pmatrix} 2\cos\phi \\ 2\sqrt{3}\sin\phi \\ 0 \end{pmatrix} \quad , \quad \phi \in [0, 2\pi] \, .$$

[**5**]

Then, using the definition of a line integral, we can compute

$$\begin{aligned} \int_C \vec{V} \cdot d\vec{r} &= \int_0^{2\pi} \begin{pmatrix} -2\sqrt{3}\sin\phi \\ 2\cos\phi \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -2\sin\phi \\ 2\sqrt{3}\cos\phi \\ 0 \end{pmatrix} d\phi \\ &= 4\sqrt{3} \int_0^{2\pi} (\sin^2\phi + \cos^2\phi) d\phi \\ &= 8\pi\sqrt{3} \end{aligned}$$

[**5**]

Remark: the opposite answer is also accepted as the orientation of C has not been specified.

6. (**Unseen**)

This question is based on the use of Stokes theorem to transform a line integral around a closed contour into a surface integral. The curl of the vector field being constant, this essentially amounts to computing the surface area of the ellipse here. So if the students remember the formula, they can use directly. Note that the same exercise was done in class (with a slightly different ellipse).

(a) Let $\overrightarrow{V}(x, y, z) = y(z-2)\vec{i} + z(x-2)\vec{j} + x(y-2)\vec{k}$. A direct computation gives $\overrightarrow{\nabla} \times \overrightarrow{V} = 2\vec{i} + 2\vec{j} + 2\vec{k}$. [**4**]

(b) Stokes theorem: Let $\overrightarrow{V}$ be a vector field with continuous partial derivatives over a surface S whose boundary is a closed curve Γ. Then

$$\int_\Gamma \overrightarrow{V} \cdot d\vec{r} = \iint_S (\overrightarrow{\nabla} \times \overrightarrow{V}) \cdot \overrightarrow{dS}\,.$$

[**2**]

(c) Here the given ellipse plays the role of Γ in the theorem and lies in the plane $z = -2$ parallel to xOy. The surface S can be taken to the flat surface in this plane enclosed by the ellipse. In particular, this means that the surface element vector $\overrightarrow{dS}$ only has a nonzero component along $\vec{k}$: $\overrightarrow{dS} = de\vec{k}$. This means that the surface integral reduces to

$$\iint_S 2\, dS = 2 \iint_S dS\,.$$

So the line integral we are after is simply twice the area of the surface enclosed by the ellipse. The parameters of the latter are $a = 4$ and $b = 3$ so using the formula for the area $A = \pi ab$, we get finally

$$\int_\Gamma \overrightarrow{V} \cdot d\vec{r} = 24\pi\,.$$

Note that this question can also be done in a more pedestrian way by using a parametrization for the interior of the ellipse and computing explicitely the surface integral or even by computing the line integral directly. The students should go for the solutions presented here as we have discussed a very similar example in class. [**14**]

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