City University
London

MA2603: Mathematical Methods

Time allowed: 3 hours

Full marks may be obtained for correct answers to FIVE of the EIGHT questions. If more than FIVE questions are answered, the best FIVE marks will be credited.

Turn over ...

1. Let f be the 2π periodic function defined on $[-\pi, \pi)$ by

$$f(x) = \sin\frac{x}{2}\,.$$

   (a) Find the Fourier coefficients of f.

   (b) Show that for all $x \in (-\pi, \pi)$,

$$\sin\frac{x}{2} = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n n}{1-4n^2}\sin nx\,.$$

   (c) By considering the point $x = \frac{\pi}{2}$, deduce that

$$\sum_{p=0}^{\infty}\frac{(-1)^{p+1}(2p+1)}{1-4(2p+1)^2} = \frac{\pi}{8\sqrt{2}}\,.$$

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2. Consider the surface of that part of the cone $y^2 = x^2 + z^2$ which lies inside the cylinder $x^2 + z^2 = 2x$.

   (a) Giving reasons, choose which of the following parametrizations is appropriate for this surface

$$\begin{array}{rcl} \overrightarrow{F}_{\pm} \quad [0,2]\times[-\frac{\pi}{2},\frac{\pi}{2}] & \rightarrow & \mathbb{R}^3 \\ (\rho,\theta) & \mapsto & \rho\cos\theta\,\vec{i} + \rho\sin\theta\,\vec{j} \pm \rho\,\vec{k} \end{array}$$

$$\begin{array}{rcl} \overrightarrow{G}_{\pm} \quad [0,2\cos\theta]\times[-\frac{\pi}{2},\frac{\pi}{2}] & \rightarrow & \mathbb{R}^3 \\ (\rho,\theta) & \mapsto & \rho\cos\theta\,\vec{i} \pm \rho\,\vec{j} + \rho\sin\theta\,\vec{k} \end{array}$$

$$\begin{array}{rcl} \overrightarrow{H}_{\pm} \quad [0,2\cos\theta]\times[-\frac{\pi}{2},\frac{\pi}{2}] & \rightarrow & \mathbb{R}^3 \\ (\rho,\theta) & \mapsto & \cos\theta\,\vec{i} + \sin\theta\,\vec{j} \pm \rho\,\vec{k} \end{array}$$

   where the subscript $\pm$ refers to the two symmetric parts of the surface.

   (b) Using the previous parametrization, show that the surface area is

$$S = 2\pi\sqrt{2}\,.$$

Turn over ...

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3. (a) Using cylindrical coordinates, show that a parametrization of the closed curve C which is the intersection of the paraboloid $4z = x^2+y^2$ with the plane $y = z$ is

$$\vec{r}(\theta) = 2\sin 2\theta\,\vec{\imath} + 4\sin^2\theta\,(\vec{\jmath}+\vec{k}) \quad , \quad \theta \in [0,\pi]\,.$$

(b) Let $\overrightarrow{V}(x,y,z) = -\frac{x}{2}\,\vec{\imath} + (1-\frac{y}{2})\,\vec{\jmath}$ be a vector field. Compute the line integral

$$I = \int_C \overrightarrow{V}\cdot d\vec{r}\,.$$

(c) Show that $\overrightarrow{V} = \overrightarrow{\nabla}\phi$ for a scalar field ϕ that you should find. Hence, deduce the value of I and check it against your previous result.

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4. Let $\overrightarrow{V}(x,y,z) = x\,\vec{\imath}+y\,\vec{\jmath}+z\,\vec{k}$ be a vector field. Let S be the surface of the cylinder $\frac{(x-1)^2}{4}+\frac{y^2}{9} = 1$ between the $z=0$ plane and the plane $3x-z=1$.

(a) Compute $\overrightarrow{\nabla}\cdot\overrightarrow{V}$.

(b) Giving reasons, choose which of the following parametrizations describes the volume inside the cylinder and the two planes.

$$\begin{aligned}\overrightarrow{F}:\ [0,1]\times[0,2\pi]\times[0,\rho^2] &\rightarrow \mathbb{R}^3\\ (\rho,\theta,z) &\mapsto (1+2\rho\cos\theta)\,\vec{\imath}+(3\rho\sin\theta)\,\vec{\jmath}+z\,\vec{k}\end{aligned}$$

$$\begin{aligned}\overrightarrow{G}:\ [0,6]\times[0,2\pi] &\rightarrow \mathbb{R}^3\\ (\rho,\theta) &\mapsto \rho\cos\theta\,\vec{\imath}+\rho\sin\theta\,\vec{\jmath}+\rho^2\vec{k}\end{aligned}$$

$$\begin{aligned}\overrightarrow{H}:\ [0,1]\times[0,2\pi]\times[0,z_0(\rho,\theta)] &\rightarrow \mathbb{R}^3\\ (\rho,\theta,z) &\mapsto (1+2\rho\cos\theta)\,\vec{\imath}+(3\rho\sin\theta)\,\vec{\jmath}+z\,\vec{k}\end{aligned}$$

where $z_0(\rho,\theta) = 6\rho\cos\theta+2$.

(c) Show that the surface integral

$$I = \iint_S \overrightarrow{V}\cdot d\vec{S}\,,$$

equals 36π.

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5. Define $\gamma = \frac{3}{2} + i\beta$, $\beta \in \mathbb{R}$ and let

$$f(z) = (\gamma z + \bar{\gamma}\bar{z})^2 + 2i(\bar{\gamma}z + \gamma\bar{z}),$$

where $\bar{z}$ (respectively $\bar{\gamma}$) is the complex conjugate of z (respectively γ).

(a) Find the real and imaginary parts of f in terms of β, x and y where $z = x + iy$.

(b) For which values of β is f differentiable? For these values, find where it is differentiable.

(c) Is f analytic anywhere?

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6. Let f be defined by

$$f(z) = \frac{1}{2z^2 - (1+2i)z + i}.$$

(a) Find the poles z_1 and z_2 of f such that $|z_1| < |z_2|$.

(b) Give the series expansion of f in the following three regions, stating in each case if it is a Taylor or Laurent series.

i. $|z| < |z_1|$.

ii. $|z_1| < |z| < |z_2|$.

iii. $|z - z_1| < |z_2 - z_1|$.

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7. Explaining your method, compute

(a)

$$I = \int_0^{2\pi} \frac{d\theta}{\cos\theta + 3\sin\theta - i}.$$

(b)

$$J = \int_{-\infty}^{\infty} \frac{x^2 dx}{1 + x^4}.$$

For the integral J, simplify your answer until you get an expression involving real numbers only.

Turn over ...

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8. In this question we compute

$$K = \int_0^\infty \frac{dx}{1+x^5}\,,$$

using the following contour C made of three pieces

$$C_x = [0, R]$$

$$C_R = \{z \in \mathbb{C}, z = R\,e^{i\theta}, 0 \le \theta \le \frac{2\pi}{5}\}$$

$$C_L = \{z \in \mathbb{C}, z = r\,e^{i\frac{2\pi}{5}}, 0 \le r \le R\}$$

in the limit $R \to \infty$. ($\Re(z)$ is the real part of z and $\Im(z)$ its imaginary part.)

(a) Sketch the contour. Let $f(z) = \frac{1}{1+z^5}$. Identify the pole of f that lies inside C when $R > 1$. Call it z_0.

(b) Show that

$$\oint_C f(z)\,dz = \int_0^R \frac{dx}{1+x^5} + \int_0^{\frac{2\pi}{5}} \frac{iRe^{i\theta}d\theta}{1+R^5e^{i5\theta}} - e^{i\frac{2\pi}{5}} \int_0^R \frac{dr}{1+r^5}\,.$$

(c) Given that $\lim_{R\to\infty} \int_0^{\frac{2\pi}{5}} \frac{iRe^{i\theta}d\theta}{1+R^5e^{i5\theta}} = 0$, deduce that

$$(1 - e^{i\frac{2\pi}{5}}) \int_0^\infty \frac{dx}{1+x^5} = 2i\pi\, Res(f, z_0)\,.$$

Hence find K (simplify your answer until you get an expression involving real numbers only).

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Internal Examiner: Dr V. Caudrelier
External Examiner: ??