UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 31 May 2007 1.30 to 4.30

PAPER 17

TORIC GEOMETRY

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let σ be a rational polyhedral cone (with a vertex) in $N_{\mathbf{R}} = N \otimes \mathbf{R}$, where N denotes a lattice in $\mathbf{R}^n$, and let $\check{\sigma}$ be the dual cone in $M_{\mathbf{R}}$, where M denotes the dual lattice. Show that the affine toric variety $X_{\check{\sigma}}$, defined over some algebraically closed field k, contains an n-dimensional algebraic torus $X_M \cong (k^*)^n$, whose natural action on itself extends to an action on $X_{\check{\sigma}}$. Prove that

$$X_{\check{\sigma}} \setminus X_M \;=\; \bigcup\{X_\nu \,:\, \nu \text{ a proper face of } \check{\sigma}\}.$$

If $\mu \in M \cap \check{\sigma}$, show that the subvariety of $X_{\check{\sigma}}$ defined by the vanishing of the monomial $\mathbf{x}^\mu$ may be identified as

$$\bigcup\{X_\nu \,:\, \nu \text{ a face of } \check{\sigma},\ \mu \notin \nu\}.$$

Describe, with proof, the order reversing bijection between the faces of σ and the faces of $\check{\sigma}$. If τ is a face of σ, show that $X_{\tau^\perp} \subset X_{\check{\sigma}}$ is an orbit under the torus action, and that the closure F_τ of $X_{\tau^\perp}$ in $X_{\check{\sigma}}$ is the affine toric variety $X_{\check{\sigma}\cap\tau^\perp}$. For τ a face of σ, find an expression for $X_{\check{\sigma}} \setminus X_{\check{\tau}}$ in terms of a union of closed sets F_γ, as γ ranges over certain faces on σ.

2 For Δ an integral polytope in $M_{\mathbf{R}} \cong \mathbf{R}^n$ (not necessarily containing the origin), describe the construction of the complete toric variety $\mathbf{P}_\Delta$. Given positive (coprime) integers $q_1, \ldots, q_n$, let $\Delta \subset \mathbf{R}^n$ be the simplex with vertices at the origin and $a_1 e_1, \ldots, a_n e_n$, where $e_1, \ldots, e_n$ is the standard basis for the lattice $M = \mathbf{Z}^n$, and $a_i = \prod_{j\neq i} q_j$ for $1 \le i \le n$. Find an explicit description for $\mathbf{P}_\Delta$ as a toric variety $X_{\Sigma,N'}$, where Σ is the standard fan defining projective n-space and $N' \supset \mathbf{Z}^n$ is an appropriate rank n lattice.

Given a general complete toric variety X_Σ, describe the construction of the *toric homogeneous coordinate ring* S, graded by elements of the class group $\mathrm{Cl}(X_\Sigma)$. For the example described above, show that as graded rings, S is isomorphic to the graded subring $k[Y_0, Y_1^{q_1}, \ldots, Y_n^{q_n}]$ of $k[Y_0, Y_1, \ldots, Y_n]$, equipped with the usual grading. Assuming that the field k has characteristic zero, explain *briefly* why, in this case, the toric variety $X_{\Sigma,N'} = \mathbf{P}_\Delta$ is the quotient of $\mathbf{A}^{n+1} \setminus \{0\}$ by an action of k^*, given by

$$(\lambda, (y_0, y_1, \ldots, y_n)) \mapsto (\lambda y_0, \lambda^{q_1} y_1, \ldots, \lambda^{q_n} y_n).$$

3 Let X_Σ be a smooth toric surface, and let $F \cong \mathbf{P}^1$ correspond to some ray in $\Sigma^{(1)}$. Show that F is the exceptional curve resulting from the toric blow-up of a smooth toric surface (i.e. F may be torically blown down to a smooth point) if and only if $F^2 = -1$.

Show that any affine toric surface $X_{\check{\sigma}}$, defined over an algebraically closed field of characteristic zero, is a quotient $\mathbf{A}^2/\langle\zeta\rangle$, where ζ is a primitive rth root of unity for some $r > 1$, and where the action of ζ is given by

$$(x, y) \mapsto (\zeta x, \zeta^q y),$$

for some $0 < q < r$ with $(q, r) = 1$. Describe (with proof) the algorithm for finding a minimal desingularization of $X_{\check{\sigma}}$ by means of continued fractions. Show that the exceptional curves F in this desingularization have $F^2 \leq -2$, and that they all have $F^2 = -2$ if and only if $q = r - 1$.

Consider now the fan Σ in $\mathbf{R}^2$ given by cones

$$\sigma_1 = \mathbf{R}_+(0, 1) + \mathbf{R}_+(r, 1) \quad \text{and} \quad \sigma_2 = \mathbf{R}_+(r, 1) + \mathbf{R}_+(1, 0).$$

Let Σ' denote the fan obtained from Σ by taking the minimal desingularization of $X_{\check{\sigma}_1}$ as above. Show that the composite morphism $X_{\Sigma'} \to X_\Sigma \to \mathbf{A}^2$ may be factored into the composite of r toric morphisms, each of which blows down a curve to a smooth point.

4 Let $\Sigma \subset N_{\mathbf{R}}$ (where N denotes a lattice in $\mathbf{R}^n$) be a fan which is complete and basic, with X_Σ denoting the corresponding smooth toric variety. Suppose that $\Sigma^{(1)} = \{\mathbf{R}_+e_1, \ldots, \mathbf{R}_+e_d\}$, for primitive vectors e_j in N, and let $F_1, \ldots, F_d$ denote the corresponding prime divisors on X_Σ. Show that any divisor D on X_Σ is linearly equivalent to one of the form $\sum_{i=1}^d a_i\, F_i$.

For $D = \sum_{i=1}^d a_i\, F_i$, identify the associated vector space $\mathcal{L}(D)$. Prove that the complete linear system $|D|$ is without fixed points if and only if the following condition holds:

For all $\sigma \in \Sigma^{(n)}$, there exists $m(\sigma)$ in the dual lattice M such that $\langle m(\sigma), e_j\rangle \geq -a_j$ for all j, with equality if $e_j \in \sigma$.

State (without proof) an analogous criterion for D to be ample.

Suppose we are given $D = \sum_{i=1}^d a_i\, F_i$ as above, which is *ample* on X_Σ. Suppose that $\sigma = \mathbf{R}_+e_1 + \ldots + \mathbf{R}_+e_n$ is a cone in $\Sigma^{(n)}$, and set $\tilde{e} = \sum_{i=1}^n e_i$ (a primitive vector in N). We make the barycentric subdivision Σ' of Σ at $\tilde{e}$, in which σ is replaced by n-dimensional cones $\sigma_1, \ldots, \sigma_n$, each σ_i containing $\tilde{e}$ and a codimension one face of σ. Let $\tilde{F}$ denote the prime divisor on $X_{\Sigma'}$ corresponding to $\tilde{e}$, and $F'_1, \ldots, F'_d$ denote those corresponding (respectively) to $e_1, \ldots, e_d$. If $a = \sum_{i=1}^n a_i$, and we set

$$D' = \sum_{i=1}^d a_i\, F'_i + a\tilde{F},$$

show that $cD' - \tilde{F}$ is ample for all sufficiently large integers c.

END OF PAPER