MATHEMATICAL TRIPOS Part III

Monday 9 June 2008 9.00 to 12.00

PAPER 9

TOPOLOGICAL GROUPS

Attempt no more than **THREE** *questions.*

There are **FOUR** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define a topological group and give, with proof, necessary and sufficient conditions in terms of neighbourhood bases for a topology on a group to give a topological group.

Which of the following statements are always true for a topological group G and which may be false? In each case give a proof or counterexample.

(a) A closed subgroup is open.

(b) An open subgroup is closed.

(c) The connected component of the identity is a subgroup which is both open and normal.

(d) If G is Hausdorff, then G is locally compact. (Hint. Consider l^2 or $\mathbb{Q}$.)

(e) If G is locally compact, it has a σ-compact open subgroup.

(f) If G is not Hausdorff it must be compact.

(g) If G has a left invariant metric (that is to say G has a left invariant metric which induces the topology) then it has a right invariant metric. (You may quote major theorems.)

(h) If G has a left invariant metric d and a right invariant metric d', then we can find a $K > 0$ such that $K^{-1}d'(x,y) \leqslant d(x,y) \leqslant Kd'(x,y)$.

2 Show that any compactly generated metrisable group has a Haar measure.

[You may assume the existence of a countable set of functions with the properties required by your proof.]

3 Let G be a locally compact Abelian Hausdorff group. Show that the multiplicative linear functionals on $L^1(G)$ with convolution may be bijectively identified with maps $f \to \hat{f}(\chi)$ where the χ are the characters of G.

Show that, if we give the group $\hat{G}$ of characters the appropriate Gelfand topology, that topology is generated by the neighbourhood basis

$$\{\chi \in \hat{G} \,:\, |\chi(x) - \gamma(x)| < \epsilon \text{ for all } x \in K\}$$

with K compact in G and $\epsilon > 0$.

4 Let G be a locally compact Abelian Hausdorff group. State Bochner's theorem and use it to prove an inversion formula of the form

$$f(x) = \int_{\hat{G}} \hat{f}(\chi)\langle x, \chi\rangle \, dm_{\hat{G}}(\chi)$$

for an appropriate measure and a reasonably wide class of f (to be specified).

Use your result to extend the notion of a Fourier transform to a linear isometry

$$\mathcal{F} : L^2(G) \to L^2(\hat{G}).$$

END OF PAPER