UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 12 June 2001 9 to 12

PAPER 5

TOPICS IN REPRESENTATION THEORY

Candidates should attempt ***ALL*** *the questions.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Let $\mathfrak{g}$ be a semisimple Lie algebra,

and $L(\lambda)$ the irreducible representation of $\mathfrak{g}$ with highest weight λ.

State the Weyl character formula, briefly explaining the definition of each term in it.

(b) Assuming the Weyl character formula, prove the Weyl dimension formula.

(c) Compute the character of the irreducible representation $L(\rho)$.

2 (a) Define the Lie algebra $\mathfrak{so}_5$, and write down a maximal torus and the root space decomposition for $\mathfrak{so}_5$.

(b) Let R be the roots of $\mathfrak{so}_5$. Draw them, and identify a set of simple roots and positive roots. Describe the Weyl group explicitly.

Describe the root lattice Q, weight lattice P, fundamental weights $\{\Lambda_i\}$, and cone P^+ of dominant weights explicitly.

(c) Compute the dimension of each irreducible representation of $\mathfrak{so}_5$.

(d) Draw the crystal for the irreducible representations $L(\Lambda_i)$, where Λ_i are the fundamental weights of $\mathfrak{so}_5$.

Draw the crystal for the adjoint representation.

(Justify your answers.)

3

Let $\mathbb{C}^3$ be the standard representation of $\mathfrak{sl}_3$, and $(\mathbb{C}^3)^*$ its dual.

What are the highest weights of these representations?

Let B and $B^\vee$ be the crystals of $\mathbb{C}^3$ and $(\mathbb{C}^3)^*$ respectively. Draw the crystals for $B \otimes B$ and $B \otimes B^\vee$.

Write down the highest weights of each of the representations that occur in these tensor products.

4

Using the representation theory of $\mathfrak{sl}_2$, prove that

$$\begin{bmatrix} n \\ k \end{bmatrix}$$

is a unimodal polynomial.

Here $\begin{bmatrix} n \\ k \end{bmatrix} = \dfrac{[n]!}{[k]![n-k]!}$, where $[n]! = \dfrac{q^n - q^{-n}}{q - q^{-1}} \quad \cdots \quad \dfrac{q^2 - q^{-2}}{q - q^{-1}} \quad \dfrac{q - q^{-1}}{q - q^{-1}}$