UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 10 June, 2005 9 to 12

PAPER 1

TOPICS IN GROUP THEORY

*Attempt **THREE** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let G be a group, let P be a Sylow p-subgroup of G. Show that if $|P : P \cap Q| \geq p^a$ for all Sylow p-subgroups Q of G different from P, then the number n_p of Sylow p-subgroups of G is congruent to 1 modulo p^a.

Let G be a simple group of order $2^e \cdot 3 \cdot 5$. Show that G has a subgroup of order 2^e or 2^{e-1} with normalizer of index 5 in G. Deduce that G is isomorphic to A_5.

Show that the group $SL_2(5)$ has order 120 and its centre Z has order 2. Show that the quotient $PSL_2(5)$ is isomorphic to A_5. Prove that $SL_2(5)$ has a unique element of order 2. [Note that any element of order 2 can be diagonalized.] Deduce that $SL_2(5)$ has no subgroup of index 2.

2 Prove that A_n is simple for $n \geq 5$.

Show that A_6 has an automorphism which is not induced by conjugation by any element of S_6.

Show that Aut A_n is isomorphic to S_n for $n \geq 5$, unless $n = 6$. [Bochert's bound on the order of primitive group can be used without proof.]

3 Prove that a minimal normal subgroup K of the finite group G is a direct product of isomorphic simple groups (possibly cyclic of prime order). Deduce that a maximal subgroup G of a non-abelian simple group X is the normalizer in X of a subgroup K of this form.

Prove that A_5 has three conjugacy classes of maximal subgroups, consisting of subgroups of orders $12, 10$ and 6, respectively.

Prove that $GL_3(2)$ has three conjugacy classes of maximal subgroups, two of order 24 and one of order 21.

[You may use without proof that both A_5 and $GL_3(2)$ are simple. Note that they contain no proper non-abelian simple subgroups.]

4 Define the transfer homomorphism. Prove that your definition is independent of the choice of transversal, and the mapping obtained is in fact a homomorphism.

State and prove the Burnside Transfer Theorem.

Let G be a primitive permutation group of degree p, where p is a prime number of the form $p = 2q + 1$, with q a prime. Show that if G contains no odd permutations, then G is simple, or G has order pq.

[Use the Frattini argument. Note that the normalizer in A_p of a Sylow p-subgroup has order pq.]

5 Write an essay on series in finite groups and some of the properties defined in terms of series. Include some proofs.

6 What is a Hall subgroup of a finite group? State the Theorem of P. Hall on Hall subgroups in finite soluble groups, and prove the existence part.

What is a Sylow basis of a finite group? Prove the existence and conjugacy of Sylow bases in finite soluble groups.

How many Sylow bases are there in S_4? And in the semidirect product $5^4 \rtimes S_4$ of an elementary abelian 5-subgroup of order 5^4 by S_4, with the action of S_4 induced by the natural permutation action on a basis of 5^4? Justify your answers.

END OF PAPER