UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 30 May 2002 9 to 12

PAPER 1

TOPICS IN GROUP THEORY

Attempt ***THREE*** *questions*

There are ***five*** *questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Prove the Theorem of Jordan that a primitive permutation group of degree n containing a 3-cycle contains the alternating group A_n.

Use this to classify the maximal subgroups of the symmetric group S_n containing a 3-cycle.

Give an example to show that for every prime p greater than 3, there exists a primitive permutation group of degree $p+1$ which contains a p-cycle but does not contain A_{p+1}.

Give an example of an odd prime p and a primitive permutation group of degree $p+1$ with order not divisible by p; justify your claim carefully.

[*You may wish to consider a certain action of* $A_5 \times A_5$ *of degree 60.*]

2 Write an essay on series in finite groups.

3 Let G be a finite group, π a set of primes. What does it mean for the subgroup H of G to be a Hall π-subgroup?

State and prove the Theorem of P. Hall concerning Hall subgroups in a finite soluble group G.

Show that the group $GL_3(2)$ has two conjugacy classes of subgroups of index 7, one consisting of stabilizers of 1-subspaces and the other of stabilizers of 2-subspaces in the 3-dimensional vector space $V_3(2)$, on which $GL_3(2)$ acts naturally.

Show further that $GL_3(2)$ has no subgroup of index 3.

[*The simplicity of* $GL_3(2)$ *may be used without proof.*]

4 Define the transfer homomorphism

$$V : G \rightarrow H/H' ,$$

where H is a subgroup of the finite group G and H' is the derived subgroup of H, and prove that it is a well-defined homomorphism.

Prove the Burnside Transfer Theorem: If a Sylow p-subgroup P of the finite group G lies in the centre of its normalizer in G, then G has a normal p-complement, that is, there is a normal subgroup K of index $|P|$ in G.

Deduce that if the Sylow p-subgroup of G is cyclic for the smallest prime p dividing the order of G, then G has a normal p-complement.

Deduce further that if G is a finite non-abelian simple group and p is the smallest prime dividing its order, then either p^3 divides $|G|$, or $p = 2$ and 12 divides $|G|$. Give infinitely many examples where the latter but not the former conclusion holds.

[*You do not need to prove here that your examples are simple.*]

5 Prove that $SL_n(q)$ (with $n \geq 2$) is generated by transvections. Deduce that $SL_n(q)$ is perfect, unless $n = 2$ and $q \leq 3$.

Use Iwasawa's Lemma (which should be stated but need not be proved) to show that $PSL_n(q)$ is simple for $n \geq 2$, unless $n = 2$ and $q \leq 3$.

Show that $PSL_2(4)$ and $PSL_2(5)$ are both isomorphic to A_5.

Show that $PSL_4(2)$ and $PSL_3(4)$ have the same order but are not isomorphic.

[*Consider the centres of their Sylow 2-subgroups.*]