UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 4 June 2001 1.30 to 4.30

PAPER 3

TOPICS IN GROUP THEORY

*Attempt **THREE** of the six questions,*
*at least **ONE** of which should be from Section B.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1 Write an essay on nilpotent groups. You should include a proof that a finite group is nilpotent if and only if it is a direct product of groups of prime power order.

2 Write an essay on soluble groups. You should include a proof that if π is a set of primes and G is a finite soluble group, then G contains a Hall π-subgroup.

3 (a) Prove that the symmetric group S_n has no outer automorphism if $n \neq 6$.

(b) Show that S_6 has an outer automorphism.

SECTION B

4 (a) Let G be a regular permutation group on a finite set Ω. Show that the centraliser of G in the symmetric group $\mathrm{Sym}(\Omega)$ is isomorphic to G.

(b) Show that the normaliser of G in $\mathrm{Sym}(\Omega)$ is isomorphic to the semi-direct product of G by its automorphism group.

(c) Now suppose that H is a finite doubly transitive permutation group with a regular normal subgroup N. Prove that N is elementary abelian. Deduce that H is isomorphic to a subgroup of an affine group $AGL(d,p)$, where p is a prime number.

5 (a) Let G be a group acting primitively on a set Ω and suppose that G is generated by $\{A^g : g \in G\}$, where A is an abelian normal subgroup of G_α for some $\alpha \in \Omega$. If N is a normal subgroup of G, show that either $G' \leq N$ or $N \leq G_{[\Omega]}$ (where $G_{[\Omega]}$ is the pointwise stabiliser of Ω).

(b) Prove that the special linear group $SL(n,F)$ of dimension $n \geq 2$ over a field F is generated by transvections.

(c) Prove that $PSL(n,F)$ is simple if $n \geq 3$ or if $n = 2$ and $|F| > 3$.

6 (a) Define the symplectic group $Sp(2m, q)$ and show that

$$|Sp(2m, q)| = q^{m^2} \prod_{i=1}^{m} (q^{2i} - 1).$$

(b) Let $\Omega = \{1, 2, \ldots, n\}$ and let V be the set of all subsets of Ω. With addition defined by symmetric difference,

$$A + B = (A \setminus B) \cup (B \setminus A),$$

V can be regarded as a vector space over the field F_2 of order 2. Show that the map $(\ , \ ) : V \times V \to F_2$ defined by

$$(A, B) = |A \cap B| \pmod 2,$$

is a symmetric bilinear form on V, preserved by the natural action of the symmetric group S_n.

If n is even, show that $(\ , \ )$ induces a non-singular alternating bilinear form on $< \Omega >^{\perp} / < \Omega >$.

(c) Prove that $Sp(4, 2) \cong S_6$.