UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 2 June, 2006 1.30 to 4.30

PAPER 7

TOPICS IN FOURIER ANALYSIS

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Show that, if $f : \mathbb{T} \to \mathbb{C}$ is continuous, then

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) r^{|n|} e^{int} = f * P_r(t)$$

where

$$P_r(t) = \frac{1 - r^2}{1 - 2r\cos t + r^2}.$$

By using the proof of Fejér's theorem as a model, or otherwise, show that

$$f * P_r(t) \to f(t)$$

uniformly as $r \to 1-$.

(ii) If p is a prime of the form $4k+1$, then it can be shown that there exists an integer u with $u^2 \equiv -1 \bmod p$. By considering the lattice

$$\Lambda = \{(n, m) \in \mathbb{Z}^2 \, : \, m \equiv nu \bmod p\},$$

show, carefully stating any theorems that you use, that p is the sum of two squares.

2 Prove the theorem of Kahane and Katznelson which states that, given any subset E of $\mathbb{T}$ of Lebesgue measure zero, there exists an $f \in C(\mathbb{T})$ whose Fourier sum diverges on E.

3

(i) Show that, if $f : \mathbb{R} \to \mathbb{C}$ is reasonably behaved, then

$$\frac{\int_{-\infty}^{\infty} x^2 |f(x)|^2 \, dx}{\int_{-\infty}^{\infty} |f(x)|^2 \, dx} \times \frac{\int_{-\infty}^{\infty} \lambda^2 |\hat{f}(\lambda)|^2 \, d\lambda}{\int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2 \, d\lambda} \geqslant \frac{1}{4}.$$

You should quote carefully any theorems that you use. Explain, with proof, when equality occurs.

(ii) Show how, if $f : \mathbb{R} \to \mathbb{C}$ is reasonably behaved and $\hat{f}(\lambda) = 0$ for $|\lambda| \geqslant \pi$, we can recover $f(t)$ for all $t \in \mathbb{R}$ from the values of $f(n)$ with n an integer.

Show that, if $\epsilon > 0$, the condition $\hat{g}(\lambda) = 0$ for $|\lambda| \geqslant \pi + \epsilon$, does not imply that g is uniquely determined by the values of $g(n)$ with n an integer.

4

(i) Prove the following version of the Riemann–Lebesgue lemma. If $f : \mathbb{T} \to \mathbb{C}$ is continuous, then $\hat{f}(r) \to 0$ as $|r| \to \infty$.

Show also that, if $k(r) > 0$ and $k(r) \to \infty$ as $r \to \infty$, then there exists a continuous function $g : \mathbb{T} \to \mathbb{C}$ such that $\limsup_{r\to\infty} k(r)|\hat{g}(r)| = \infty$.

(ii) In this question we use the normalisation of Haar functions such that

$$\int |H(t)|^2 \, dt = 1$$

for each Haar function H. Prove the case of Tao's theorem which states that, if $f : \mathbb{T} \to \mathbb{C}$ is continuous, then

$$\sum_{|\hat{f}(H)| \geqslant \delta} \hat{f}(H)H(t) \to f(t)$$

uniformly as $\delta \to 0+$.

END OF PAPER