UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 6 June, 2006 9 to 12

PAPER 11

TOPICS IN BANACH SPACES

Attempt ***FOUR*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Prove that every normalized basic sequence has a subsequence that generates a spreading model. [*You may use Ramsey's Theorem without proof if you state it clearly.*]

(b) Show that a spreading model generated by a normalized weakly null basic sequence is 1-suppression-unconditional.

(c) Explain briefly why every infinite-dimensional Banach space has an unconditional spreading model.

2 (a) Define hereditarily indecomposable Banach spaces.

(b) Let $T: X \to X$ be a bounded, linear map on a hereditarily indecomposable Banach space X. Assume that for every finite-codimensional subspace Y of X and $\varepsilon > 0$ there exists $y \in Y$, $||y|| = 1$ such that $||T(y)|| < \varepsilon$. Explain briefly (without proof) how to construct, for given $\varepsilon_i > 0$ ($i \in \mathbb{N}$), a normalized basic sequence (e_i) in X such that $||T(e_i)|| < \varepsilon_i$ for all $i \in \mathbb{N}$. Deduce that T is strictly singular, i.e. for every infinite-dimensional subspace Y of X and $\varepsilon > 0$ there exists $y \in Y$, $||y|| = 1$ such that $||T(y)|| < \varepsilon$.

(c) State and prove Gowers' Dichotomy Theorem. [*You may use Gowers' Ramsey Theorem for Banach spaces without proof if you state it clearly. You may assume standard results about bases.*]

3 State and prove Rosenthal's ℓ_1-theorem. [*You may use results from infinite Ramsey theory without proof if you state them clearly.*]

4 (a) Show that a well-founded, closed tree in a Polish space has countable height.

(b) Define Bourgain's ℓ_1-index and use it to show that there is no separable, reflexive space that is universal for the class of all separable, reflexive spaces.

5 Let X be a Banach space with a basis and with norm $|| \cdot ||$. Let us say that X has bounded distortions if there exists $D > 0$ such that for every block subspace Y of X and for every equivalent norm $||| \cdot |||$ on Y there is a block subspace Z of Y on which $|| \cdot ||$ and $||| \cdot |||$ are D-equivalent.

Show that a space with bounded distortions contains an unconditional basic sequence. [*You may use the result that a well-founded, closed tree in a Polish space has countable height.*]

6 (a) Define Ramsey subsets of $\mathbb{N}^{(\omega)}$. Show that every open subset of $\mathbb{N}^{(\omega)}$ (in the product topology) is Ramsey.

(b) Let (x_i) be a normalized, weakly null sequence in a Banach space X. Let $x^*: \mathbb{N}^{(\omega)} \to B_{X^*}$, $M \mapsto x^*_M$ be a continuous map, where the unit ball B_{X^*} of X^* is given the weak*-topology. Let $\delta > 0$. Show that there exists $L \in \mathbb{N}^{(\omega)}$ such that for all $M \in L^{(\omega)}$ we have $|x^*_{M'}(x_m)| < \delta$, where $m = \min M$ and $M' = M \setminus \{m\}$.

END OF PAPER