MATHEMATICAL TRIPOS Part III

Friday 6 June 2008 9.00 to 12.00

PAPER 17

THE X-RAY TRANSFORM IN GEOMETRY AND DYNAMICS

Attempt no more than **THREE** *questions.*

There are **FIVE** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Let ϕ_t be a smooth flow on a closed manifold. Define the Anosov property for ϕ_t .

(b) State in terms of continuous quadratic forms, a sufficient condition for a flow to be Anosov.

(c) Show that the geodesic flow of a closed surface of negative curvature is Anosov. When the curvature is constant and equal to -1, describe explicitly the stable and unstable bundles.

2 (a) State the Livsic theorem for a transitive Anosov flow.

(b) Let M be a closed oriented surface with a Riemannian metric of negative curvature. Let SM be its unit sphere bundle, X the geodesic vector field on SM , and V the vertical vector field. Assume the integral identity

$$\int_{SM} (XVu)^2 \, d\mu - \int_{SM} K(Vu)^2 \, d\mu = \int_{SM} (VXu)^2 \, d\mu - \int_{SM} (Xu)^2 \, d\mu \, .$$

where K is the Gaussian curvature, μ the Liouville measure and $u : SM \to \mathbb{R}$ any smooth function. Show that if a 1-form θ on M integrates to zero along every closed geodesic, then θ must be exact.

(c) Does the result in the previous part hold without any curvature assumptions?

3 (a) Define the scattering relation α of a compact simple Riemannian manifold with boundary $(M, \partial M, g)$.

(b) Let g_1 and g_2 be two simple metrics on M with the same boundary distance function. Show that there exists a diffeomorphism $\psi : M \to M$, which is the identity on the boundary, such that g_1 and $\psi^* g_2$ coincide on ∂M (that is, $g_1(u,v) = \psi^* g_2(u,v)$ for all $u, v \in T_x M$ and all $x \in \partial M$).

(c) Let g_1 and g_2 be two simple metrics such that they have the same boundary distance function and they agree on ∂M . Show that g_1 and g_2 have the same scattering relation.

4 Let (M, g) be a closed oriented Riemannian surface with unit sphere bundle SM. Let $\lambda : SM \to \mathbb{R}$ be any smooth function and consider the flow ϕ_t on SM defined by the differential equation

$$\frac{D\dot{\gamma}}{dt} = \lambda(\gamma, \dot{\gamma})\, i\dot{\gamma}\,,$$

where i indicates rotation by $\pi/2$ according to the orientation of the surface and $\gamma : \mathbb{R} \to M$. Denote by F the vector field associated with ϕ_t, and let $\Lambda(SM)$ be the bundle over SM whose fibre at $(x, v) \in SM$ is given by all 2-dimensional subspaces of $T_{(x,v)}SM$ that contain $F(x, v)$.

(a) Define the Maslov cycle Λ_V.

(b) Show that the action of ϕ_t lifts to an action ϕ_t^* on $\Lambda(SM)$.

(c) If F^* denotes the vector field of ϕ_t^*, show that F^* is transversal to the Maslov cycle Λ_V.

5 (a) Let N be a closed orientable manifold and F a non-zero vector field with flow ϕ_t. Show that ϕ_t preserves a volume form if and only if the divergence of F with respect to any volume form is a coboundary.

(b) Let (M, g) be a closed oriented Riemannian surface with unit sphere bundle SM. Consider the flow ϕ_t on SM defined by the differential equation

$$\frac{D\dot{\gamma}}{dt} = \langle E(\gamma), i\dot{\gamma}\rangle\, i\dot{\gamma}\,,$$

where i indicates rotation by $\pi/2$ according to the orientation of the surface, $\gamma : \mathbb{R} \to M$ and E is a given vector field on M. As before, let F be the vector field associated with ϕ_t. Compute the divergence of F with respect to the Liouville volume form in SM.

(c) Suppose that the flow ϕ_t in the previous part is Anosov. Show that ϕ_t preserves a volume form if and only if E is the gradient of a smooth function. [*You may use results on the kernel of the X-ray transform provided they are clearly stated.*].

END OF PAPER