UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 4 June 2002 1.30 to 4.30

PAPER 76

THE LAMBDA CALCULUS

Attempt ***FIVE*** *questions*

There are ***eight*** *questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define the *diamond property* and the *Church-Rosser* property of a reduction relation. Show that one-step β-reduction does not have the diamond property. Sketch a proof that it does have the Church-Rosser property.

2 What is a fixed point combinator? Exhibit a term G of the λ-calculus such that Z is a fixed point combinator if and only if Z is a fixed point of G. Show that your G has the property claimed.

Exhibit two distinct fixed point combinators, and explain why they are distinct.

Let $\mathbf{K} \equiv \lambda xy.x$ and Z a fixed point combinator. Show that $Z\mathbf{K}$ is unsolvable.

3 What does it mean for a λ-term to be solvable? What does it mean for a λ-term to have head normal form? Show that M is solvable if and only if M has head normal form. (The Standardization Theorem may be assumed.)

A closed term E is *easy* just when for any other closed term M, the equation $E = M$ is consistent with the λ-calculus.

(i) Show that I is not easy. (You may quote any general form of Böhm's Theorem which you may require.)

(ii) Show that if S is easy then so is ST for any T. Deduce that easy terms are unsolvable.

4 Define a system of numerals giving explicit terms for sucessor, predecessor and test for zero. Prove that for your numeral system the total λ-definable functions are closed under primitive recursion.

State carefully a result concerning closure under minimization, and sketch the proof. What can you now deduce about the λ-definable functions?

5 Let $\# : \Lambda \to \mathbb{N}$ be an effective Gödel numbering of the λ-terms and let d_n be some standard numeral system. The λ-code of a λ-term M is $d(M) = d_{\#(M)}$.

(i) Explain why there are λ-terms A and C with $Ad(S)d(T) = d(ST)$ and $Cd_n = d(d_n)$.

(ii) For $F \in \Lambda$, set $H \equiv \lambda x F(Ax(Cx))$. Show that $Q \equiv Hd(H)$ satisfies $Fd(Q) = Q$.

(iii) Construct a term D such that

$$Dd_0UV = U$$
$$Dd_1UV = V$$

for all U, V.

(iv) Let $\mathcal{U}$, $\mathcal{V}$ be two non-empty disjoint collections of λ-terms closed under β-equality. Suppose there were a λ-term G with $Gd_n = d_0$ or d_1 for all n, and satisfying

$$Gd(M) = \begin{cases} d_1 & M \in \mathcal{U} \\ d_0 & M \in \mathcal{V} \end{cases} .$$

By considering $F = \lambda x.DUV(Gx)$ for fixed $U \in \mathcal{U}$, $V \in \mathcal{V}$, derive a contradiction. Is β-equality decidable?

6 Describe briefly the Böhm tree $BT(M)$ of a term M of the λ-calculus. What is a Böhm transformation? Show that for $\alpha \in BT(M)$ there is a Böhm transformation π such that M^π equals some substitution instance of M_α. Explain briefly how this result is used in the proof of Böhm's Theorem.

Let $\mathbf{K} = \lambda xy{\cdot}\alpha$ and $\mathbf{S} = \lambda xyz{\cdot}xz(yz)$. Give an explicit term M such that $M\mathbf{K} = \mathbf{T}$ and $M\mathbf{S} = \mathbf{F}$.

[Here $\mathbf{T} = \lambda xy \cdot \alpha$ and $\mathbf{F} = \lambda xy \cdot y$.]

7 What is a cpo? Let D and E be cpos. Define the function space $[D \to E]$ and show that it is a cpo. Suppose that D is a cpo with $[D \to D]$ a retract of D. Show how to define application on the elements of D so that the resulting structure is a λ-model.

Describe such a retraction giving the standard $P\omega$ model.

8 Describe the language $\Lambda\bot^{\mathbb{N}}$ and explain how its terms are interpreted in $P\omega$.

Describe the reduction of a term in $\Lambda\bot^{\mathbb{N}}$. Show that if $M^I \to M^J$ then the interpretations in $P\omega$ satisfy $[M^I] \subseteq [N^J]$. State the relation between the Böhm trees of M and N.

Prove the Approximation Theorem for the $P\omega$ model of the λ-calculus.

[You may assume that any fully indexed term has a unique normal form.]