UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 8 June 2007 9.00 to 11.00

PAPER 81

THE FLUID DYNAMICS OF SWIMMING ORGANISMS

Attempt ***TWO*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

This is an ***OPEN BOOK*** *examination.*

Candidates may bring their course handouts, including example sheets, and any handwritten material into the examination. Textbooks, laptops etc are not permitted.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A sea snake of length L has a uniform circular cross-section and swims at speed U by passing a constant-amplitude, planar wave of displacement down its body at constant speed c relative to the snake's centreline, speed $V(=\alpha c$ where $\alpha < 1)$ relative to the snake in the x-direction (unit vector $\mathbf{i}$). The force exerted on an element ds of the snake consists of two parts:

- a normal component $K_N\,|\mathbf{w}.\mathbf{n}|\,(\mathbf{w}.\mathbf{n})\,\mathbf{n}ds$
- a tangential component $K_T\,|\mathbf{w}.\mathbf{t}|^{3/2}\,\mathbf{t}s^{1/2}ds$

where $\mathbf{n}$, $\mathbf{t}$ are unit vectors normal and tangential to the centreline, $\mathbf{w}$ is the velocity of the fluid far away relative to ds, and K_N, K_T are constants.

Explain briefly the reason for each of the above terms taking the form it does in this nonlinear, high-Reynolds-number version of Resistive Force Theory, and discuss any approximations made.

Deduce that the mean swimming speed U will be given by the following equation:

$$K_N(V-U)^2\int_0^L\left|\frac{d\hat{Y}}{ds}\right|^3 ds = K_T\int_0^L\left|c-(V-U)\frac{dX}{ds}\right|^{3/2}\frac{dX}{ds}s^{1/2}ds$$

where the material point at s on the animal's centreline has position vector
$[X(s,t),\ Y(s,t)]$.

In the case for which the lateral displacement of the centreline is given in the frame of reference moving with the wave by $Y(s) = \beta\sin ks$, where $\beta k \ll 1$ and $k = 2\pi/L$, deduce that

$$4\beta^3k^2K_N(V-U)^2 \approx K_TL^{3/2}(c-V+U)^{3/2}. \qquad (1)$$

Instead of using the above model, use Lighthill's small-amplitude elongated body theory to estimate the thrust exerted by the snake. Balance this against the viscous drag, estimated using boundary-layer-theory, and show that equation (1) is replaced by:

$$\frac{3}{8}M_A\beta^2k^2(V^2-U^2) = K_T(c-V+U)^{3/2}.L^{3/2},$$

where M_A is the added mass per unit length.

Discuss some of the weaknesses of the above models.

2 A bacterium consists of a spherical body of radius a and a thin, rigid, helical flagellum of length L which rotates about its axis, which is the extension of a line through the centre of the sphere and a given point on its surface. The helix has radius b, makes an angle α with the axial direction, and has angular velocity of magnitude Ω_o relative to the body. The bacterium swims in a fluid of viscosity μ.

Use low-Reynolds-number resistive force theory to analyse the motion and show that the swimming velocity $-U\mathbf{i}$ and the angular velocity $\Omega\mathbf{i}$ of the cell body are given by the following equations:

$$(K_N \sin^2\alpha + K_T \cos^2\alpha + 6\pi\mu\frac{a}{L})U = (K_N - K_T)b(\Omega_o - \Omega)\sin\alpha\cos\alpha$$

$$bL(K_N - K_T)\sin\alpha\cos\alpha U = (K_N\cos^2\alpha + K_T\sin^2\alpha)b^2L(\Omega_o - \Omega) - 8\pi\mu\Omega a^3,$$

where $\mathbf{i}$ is the unit vector parallel to the axis of the helix, and K_N, K_T are the normal and tangential force coefficients of the flagellum.

Calculate U for the case $K_N/K_T = 2, \alpha = \pi/4, a = b$

[The viscous torque on a sphere rotating with angular velocity $\mathbf{\Omega}$ in a fluid otherwise at rest is $-8\pi\mu a^3\mathbf{\Omega}$. The details of the motion of the flagellum in the immediate vicinity of the cell body may be ignored.]

3 A vertical, cylindrical pipe of radius R contains a dilute suspension of spherical, bottom-heavy, swimming micro-organisms that are denser than the fluid in which they are swimming. The suspension flows downward in the pipe under the action of a constant effective pressure gradient

$$\frac{d\hat{p}}{d\hat{z}} = +\frac{4\mu W_o}{R^2},$$

where W_o is a velocity scale, μ is the fluid viscosity and $\hat{z}$ is measured vertically upwards. A $\hat{}$ over a quantity denotes that it is a dimensional variable. Neglecting sedimentation relative to cell swimming and assuming that the cell swimming direction is given deterministically by a viscous-gravitational torque balance, but allowing for a scalar cell diffusivity, investigate possible steady flows in which the fluid has zero radial velocity and has vertical velocity $-W_o(1-\hat{r}^2/R^2)+\hat{w}(\hat{r})$, and the cell concentration is $\hat{n}(\hat{r})$, where $\hat{r}$ is the radial coordinate. Write down and explain the governing equations and boundary conditions, including the contribution to the stress tensor arising from the cells' swimming stresslets, and show that the equations can be reduced to the following non-dimensional form:

(1) $$\frac{dn}{dr} = \chi n \sin\theta$$

(2) $$\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right) + \frac{\sigma}{r}\frac{d}{dr}(rn\sin 2\theta) = \gamma n$$

(3) $$\sin\theta = -\lambda\frac{d}{dr}(r^2+w)$$

where

$$\chi = \frac{V_c R}{D}, \quad \sigma = \frac{SN_oR}{2\mu W_o}, \quad \gamma = \frac{g\triangle\rho v N_o R^2}{\rho\mu W_o}, \quad \lambda = \frac{W_oB}{2R}$$

and V_c is the cell swimming speed, B the gyrotactic time-scale, D the cell diffusivity, S the stresslet strength of a single cell, N_o the average cell concentration, ρ the fluid's density, $\rho+\triangle\rho$ the cell's density and v the cell volume. θ is the angle between the cell swimming direction and the vertical.

You may assume that $BW_o/R < 1$ and that the $\hat{z}-$component of div $(\hat{n}\hat{\mathbf{p}}\hat{\mathbf{p}})$ is

$$\frac{1}{\hat{r}}\frac{d}{d\hat{r}}\left[\hat{r}\hat{n}(\hat{r})\sin\theta\cos\theta\right].$$

Why can the $\hat{r}$-component of the divergence of the particle stress tensor be ignored?

Analyse the case $\gamma \ll 1,\ \sigma = \gamma\sigma_1,\ \sigma_1 = O(1),\ \lambda\chi = O(1)$, by taking

$$w(r) \approx \gamma w_1(r),\ n \approx n_o(r) + \gamma n_1(r),\ \theta \approx \theta_0 + \gamma\theta_1.$$

Show that $n_o(r) = n_{oo}e^{-\lambda\chi r^2}$ where $n_{oo} = \frac{\lambda\chi}{1-e^{-\lambda\chi}}$.

Derive an equation for $\frac{dw_1}{dr}$ as a function of r and solve it in the form of an integral.

Show that the perturbation w to the pipe flow is negative unless σ_1 is sufficiently negative. Discuss this result.

4 A dilute suspension of bottom-heavy, randomly swimming microorganisms, all of which swim with constant speed V_s, is arranged in such a way that the probability distribution function (pdf) of cell orientation $\mathbf{p}$ at time $t = 0$, $f(\mathbf{p},0)$, is isotropic. We wish to calculate how the mean swimming velocity $\mathbf{V}_c$ varies with time.

Write down the Fokker-Planck equation for the *time-dependent* pdf, $f(\mathbf{p}, t)$, and show that it reduces in this case to

$$B\frac{\partial f}{\partial t} - \sin\theta\frac{\partial f}{\partial \theta} - 2\cos\theta f = \frac{BD_r}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right), \qquad (1)$$

where θ is the angle between $\mathbf{p}$ and the upwards vertical (unit vector $\mathbf{k}$), B is the gyrotactic time-scale and D_r is the rotational diffusivity, which represents the randomness of cell swimming.

(i) Find the steady-state solution of equation (1), $f_o(\theta)$,and calculate the corresponding value of $\mathbf{V}_c$.

(ii) In the case $D_r = 0$, solve equation (1), subject to the appropriate initial condition, by using the substitutions $f(\theta, t) = 1/g(x, t), x = \cos\theta$, and seeking a solution that is quadratic in x (or otherwise), to show that

$$f(\theta, t) = \frac{\alpha/\pi}{[1 + \alpha - (1 - \alpha)x]^2}$$

where $\alpha = e^{-2t/B}$. Hence find $\mathbf{V}_c$ in this case.

(iii) Verify that the limit of $\mathbf{V}_c$ from part (i) as $\lambda^{-1} = BD_r \to 0$ is the same as the limit of $\mathbf{V}_c$ from part (ii) as $t \to \infty$, despite the neglect of diffusion in the latter case. Will the time-scales for these limits to be achieved also be the same?

END OF PAPER