UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 1 June 2007 1.30 to 4.30

PAPER 52

SYMMETRY AND PARTICLE PHYSICS

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
10 sheets of triangular graph paper

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $\mathcal{L}$ be a Lie algebra, and let d be a representation of $\mathcal{L}$ acting on a vector space V. Define the Killing form κ of $\mathcal{L}$, and prove that

$$\kappa([X,Y],Z) = \kappa(X,[Y,Z])$$

for $X,Y,Z \in \mathcal{L}$. Let $c_{ab}{}^{d}$ be the structure constants of $\mathcal{L}$, and let $c_{abc} \equiv c_{ab}{}^{d}\kappa_{cd}$. Show that c_{abc} is totally antisymmetric in a,b,c.

What does it mean for $\mathcal{L}$ to be semi-simple or compact? If $\mathcal{L}$ is semi-simple, define the Casimir operator C associated with the representation d, and show that $[d(X),C]=0$ for all $X \in \mathcal{L}$.

Let n be a positive integer. Let E_{ij} denote the $n \times n$ matrix with elements 0, except for the element in the i-th row and j-th column which is 1, so that

$$(E_{ij})_{pq} = \delta_{ip}\delta_{jq} \ .$$

Let $\mathcal{M}$ denote the real Lie algebra with basis $\{E_{ij}, 1 \leqslant i,j \leqslant n\}$. Compute the structure constants $c_{ij,pq}{}^{mn}$ in this basis (where basis elements are labelled by pairs of integers ij), and compute the components $\kappa_{ij,pq}$ of the Killing form. Determine if $\mathcal{M}$ is semi-simple or compact.

Let $\mathcal{L}$ denote the vector space of real upper triangular matrices which has basis $\{E_{ij} : 1 \leqslant i < j \leqslant n\}$. Show that if $X,Y \in \mathcal{L}$ then $[X,Y] \in \mathcal{L}$. Determine $\mathcal{L}'$, where

$$\mathcal{L}' = \{X \in \mathcal{L} : \kappa(Y,X) = 0 \text{ for all } Y \in \mathcal{L}\} \ .$$

2 Let T_a be the standard basis of generators for $\mathcal{L}(SU(2))$ satisfying $[T_a, T_b] = \epsilon_{abc}T_c$, and let d be a finite-dimensional representation of $\mathcal{L}(SU(2))$ acting on a complex vector space V. Let

$$J_3 = i\,d(T_3), \qquad J_\pm = \frac{i}{\sqrt{2}}(d(T_1) \pm i\,d(T_2)) \ .$$

(i) Suppose that the representation d is irreducible. Define the highest weight j of the representation. Show that j must exist and that $2j$ is a non-negative integer. If $|j,j\rangle$ is the highest weight state, show that V is $(2j+1)$-dimensional with basis $\{(J_-)^\ell\,|j,j\rangle : \ell = 0,\ldots,2j\}$. If, in addition, d is antihermitian and $|j,j\rangle$ is unit-normalized, prove that

$$|j,m\rangle = \frac{1}{\sqrt{N_{j-m}}}(J_-)^{j-m}\,|j,j\rangle$$

for $m = -j,\ldots,j$ defines an orthonormal basis of V if

$$N_\ell = \frac{(2j)!\ell!}{2^\ell(2j-\ell)!}, \quad \ell = 0,\ldots,2j \ .$$

(ii) Suppose that d_1, d_2 are finite-dimensional antihermitian irreducible representations of $\mathcal{L}(SU(2))$ acting on complex vector spaces V_1, V_2 such that $\dim V_1 = 2j_1 + 1$, $\dim V_2 = 2j_2 + 1$. Let $V = V_1 \otimes V_2$ be the tensor product vector space, and let $d = d_1 \otimes 1 + 1 \otimes d_2$ be the tensor product representation of $\mathcal{L}(SU(2))$ acting on V. State how V decomposes into a direct sum of subspaces on which d is irreducible.

Define $|m_1 : m_2\rangle \equiv |j_1, m_1\rangle \otimes |j_2, m_2\rangle$ and suppose $|j,m\rangle$ for $m = -j,\ldots,j$ denotes a state in the $(2j+1)$-dimensional subspace of V on which d is irreducible, with $J_3\,|j,m\rangle = m\,|j,m\rangle$. Prove the recurrence relation

$$\begin{aligned}\sqrt{(j \pm m)(j \mp m + 1)}\,\langle m_1 : m_2|j, m \mp 1\rangle &= \sqrt{(j_1 \mp m_1)(j_1 \pm m_1 + 1)}\,\langle m_1 \pm 1 : m_2|j,m\rangle \\ &\quad + \sqrt{(j_2 \mp m_2)(j_2 \pm m_2 + 1)}\,\langle m_1 : m_2 \pm 1|j,m\rangle\end{aligned}$$

[*You may assume the commutation relations* $[J_3, J_\pm] = \pm J_\pm$ *and* $[J_+, J_-] = J_3$ *throughout this question.*]

3 Let $\mathcal{L}(G)$ be the matrix Lie algebra of a matrix Lie group G, and let D^f_μ be the fundamental covariant derivative which acts on column vectors χ via

$$D^f_\mu \chi = \partial_\mu \chi + A_\mu \chi$$

where $\partial_\mu = \frac{\partial}{\partial x^\mu}$, $\mu = 0, 1, 2, 3$ and A_μ is the $\mathcal{L}(G)$-valued gauge potential. If χ transforms as $\chi \to \chi' = g\chi$ where $g = g(x) \in G$, derive the transformation rule for A_μ which is necessary in order for $D^f_\mu \chi$ to transform in the same way as χ.

The Yang-Mills field strength of A_μ is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \ .$$

Determine how F transforms under gauge transformations.

Let D_μ denote the adjoint covariant derivative which acts on $\Phi \in \mathcal{L}(G)$ by

$$D_\mu \Phi = \partial_\mu \Phi + [A_\mu, \Phi] \ .$$

If A_μ transforms in the same way as found for the fundamental covariant derivative, and Φ transforms as $\Phi \to \Phi' = g\Phi g^{-1}$, prove that $D_\mu \Phi$ transforms in the same way as Φ. Also prove that

$$[D_\mu, D_\nu]\Phi = [F_{\mu\nu}, \Phi] \ .$$

Suppose that $\mathcal{L}(G)$ is semi-simple with Killing form κ. Determine the field equations associated with the Lagrangian density L, where

$$L = \frac{1}{4e^2}\kappa(F_{\mu\nu}, F^{\mu\nu}) - \frac{1}{2}\kappa(D_\mu \Phi, D^\mu \Phi) \ ,$$

and e is constant. Prove that if $A_0 = 0$, $\partial_0 \Phi = 0$ and $\partial_0 A_i = 0$ for $i = 1, 2, 3$ and

$$F_{ij} = e\epsilon_{ijk} D_k \Phi$$

then all of the field equations are satisfied.

[For this question, you may assume $\kappa(X, [Y, Z]) = \kappa([X, Y], Z)$ for $X, Y, Z \in \mathcal{L}(G)$. The Minkowski metric is $\eta = \mathrm{diag}(1, -1, -1, -1)$.]

4 Write an account of the representation theory of $\mathcal{L}(SU(3))$ and its application to the classification of light hadrons in the quark model. The essay should include a discussion of the representation theory of irreducible antihermitian representations of $\mathcal{L}(SU(3))$ using weight diagrams, describing the different types of weight diagram which can arise and stating (without proof) how the degeneracy of the states in such a representation depends on the shape of the weight diagram. You should also describe how to decompose tensor product representations using weight diagrams, and how these decompositions are used to classify light hadrons.

[*You may assume the standard non-vanishing commutation relations*

$$[H_1, E^1_\pm] = \pm E^1_\pm \ , \qquad [H_1, E^2_\pm] = \mp\frac{1}{2}E^2_\pm \ , \qquad [H_1, E^3_\pm] = \pm\frac{1}{2}E^3_\pm \ ,$$

and

$$[H_2, E^1_\pm] = 0 \ , \qquad [H_2, E^2_\pm] = \pm\frac{\sqrt{3}}{2}E^2_\pm \ , \qquad [H_2, E^3_\pm] = \pm\frac{\sqrt{3}}{2}E^3_\pm \ ,$$

and

$$[E^1_+, E^1_-] = H_1 \ , \quad [E^2_+, E^2_-] = \frac{\sqrt{3}}{2}H_2 - \frac{1}{2}H_1 \ , \quad [E^3_+, E^3_-] = \frac{\sqrt{3}}{2}H_2 + \frac{1}{2}H_1 \ ,$$

and

$$\begin{aligned}
[E^1_+, E^2_+] &= \frac{1}{\sqrt{2}}E^3_+ \ , \qquad [E^1_-, E^2_-] = -\frac{1}{\sqrt{2}}E^3_- \ , \\
[E^1_+, E^3_-] &= -\frac{1}{\sqrt{2}}E^2_- \ , \qquad [E^1_-, E^3_+] = \frac{1}{\sqrt{2}}E^2_+ \ , \\
[E^2_+, E^3_-] &= \frac{1}{\sqrt{2}}E^1_- \ , \qquad [E^2_-, E^3_+] = -\frac{1}{\sqrt{2}}E^1_+ \ ,
\end{aligned}$$

where $iH_1, iH_2, i(E^m_+ + E^m_-), E^m_+ - E^m_-$ are the antihermitian $\mathcal{L}(SU(3))$ generators in the representation d and $H_1 = d(h_1)$, $H_2 = d(h_2)$ with

$$h_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad h_2 = \begin{pmatrix} \frac{1}{2\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{2\sqrt{3}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{3}} \end{pmatrix} . \qquad]$$

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