UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 4 June 2002 1.30 to 4.30

PAPER 58

SYMMETRIES AND PATTERNS

*Attempt **THREE** questions*

*There are **four** questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider the solution of the modified Swift-Hohenburg equation

$$\frac{\partial \psi}{\partial t} = \mu\psi - \left(\frac{\partial^2}{\partial x^2} + K\right)^2 \psi + \psi^2, \ -1 < x < 1: \ \psi = \frac{\partial^2 \psi}{\partial x^2} = 0, \ x = \pm 1.$$

(i) Show that there are two families of linear eigenfunctions for all $n \geq 0$, namely

$$\psi_j(x) = \cos{(n + \tfrac{1}{2})\pi x} \ (j = 2n); \quad \psi_j(x) = \sin{(n+1)\pi x} \ (j = 2n+1).$$

which have steady-state bifurcations at $\mu = \mu_j(K)$ $(j = 0, 1, \ldots)$. Find $\mu_j(K)$ and determine μ^*, K^* such that $\mu_0(K^*) = \mu_1(K^*) = \mu^*$.

(ii) Show that the centre manifold at this point is two-dimensional, so that $\psi \approx A(t)\psi_0 + B(t)\psi_1 + \ldots$. Use group theoretical arguments to give the normal form equations. Show that the extended centre manifold, truncated at quadratic order, takes the form

$$\frac{\partial A}{\partial t} = \lambda_1 A + a_1 A^2 + a_2 B^2 \ ,$$
$$\frac{\partial B}{\partial t} = \lambda_2 B + a_3 AB$$

Determine λ_1, λ_2 as linear functions of $\mu - \mu^*$, $K - K^*$, and indicate, without detailed calculation, how a_1, a_2, a_3 might be found.

(iii) It is given that $8a_1 = 10a_2 = 5a_3 > 0$. Find all the steady state solutions of the truncated system above and compute the location of any bifurcation points in (λ_1, λ_2) space, indicating the type of each. Sketch the bifurcation diagram. Sketch also the solution curves for A as a function of λ_1, when $\lambda_1 = \lambda_2 + \Delta$ (Δ constant), distinguishing between positive and negative Δ. Show clearly any stable branches.

2 Consider a steady-state bifurcation problem which has the symmetry of the hexagonal lattice with respect to rotations and translations, but *not* reflections (this would be appropriate, for instance, for convection in a rotating layer). It may be assumed that the solutions may be taken as proportional to $Ae^{i\mathbf{k_1}\cdot\mathbf{x}} + Be^{i\mathbf{k_2}\cdot\mathbf{x}} + Ce^{i\mathbf{k_3}\cdot\mathbf{x}}$, where A, B, C are complex amplitudes and $\mathbf{k_1}, \mathbf{k_2}, \mathbf{k_3}$ are the three fundamental lattice vectors with $\mathbf{k_1} + \mathbf{k_2} + \mathbf{k_3} = 0$. Consider the possible existence of steady solution branches in the form of *hexagons* ($A = B = C$(real)) and *rolls* ($A \neq 0$(real), $B = C = 0$). Find the isotropy subgroup and the fixed point subspace for each of these solutions, and show that the fixed point subspaces are one-dimensional in each case. What does this say about the existence of the branches? Use symmetry arguments to construct the normal form equations. Show that when truncated at cubic order the normal form may be written

$$
\begin{aligned}
\frac{dA}{dt} &= \mu A + B^* C^* - \nu_1 |A|^2 A - (\nu_2 + \delta)|B|^2 A - (\nu_2 - \delta)|C|^2 A \\
\frac{dB}{dt} &= \mu B + C^* A^* - \nu_1 |B|^2 B - (\nu_2 + \delta)|C|^2 B - (\nu_2 - \delta)|A|^2 B \\
\frac{dC}{dt} &= \mu C + A^* B^* - \nu_1 |C|^2 C - (\nu_2 + \delta)|A|^2 C - (\nu_2 - \delta)|B|^2 C
\end{aligned}
$$

with real coefficients $\mu, \nu_1, \nu_2, \delta$.

Using the truncated normal form above, find conditions for the stability of both the roll and hexagon solutions as functions of $\mu, \nu_1, \nu_2, \delta$. Show in particular that when $\delta \neq 0$ and $\nu_2 > \nu_1 > 0$ the branch of hexagons can lose stability at a Hopf bifurcation, while rolls are never stable if $\delta^2 > (\nu_2 - \nu_1)^2$.

Without detailed calculation, sketch a possible phase portrait in the case that both these solution branches are unstable (you may assume that the trajectory remains in the invariant subspace in which A, B, C are real).

3 Consider the evolution equation, for real $A = A(x,t)$, $-\infty < x < \infty$

$$\frac{\partial A}{\partial t} = \mu(x,t)A + \alpha A^3 - A^5 + \frac{\partial^2 A}{\partial x^2} \qquad (*)$$

where $\alpha > 0$ is constant. This equation describes the envelope for a non-autonomous, inhomogeneous pattern-forming bifurcation.

(i) Show that when $\mu = \mu_0$ (constant) there are steady solutions satisfying $A \to 0$, $x \to -\infty$, $A^2 \to C^2 \neq 0$ as $x \to \infty$ only if $\mu_0 = -3\alpha^2/16$. Verify that in this case

$$A^2 = \frac{3\alpha}{4}\left(1 + \frac{3\alpha}{4}\exp(-\alpha\frac{\sqrt{3}}{2}(x - X_0))\right)^{-1} \equiv A_0^2(x - X_0)\,,$$

where X_0 is any constant. Show also that for any suitably well-behaved function $R(x)$ the identity

$$\int_{-\infty}^{\infty} \frac{\partial A_0}{\partial x}\left[(\mu_0 + 3\alpha A_0^2 - 5A_0^4)R + \frac{\partial^2 R}{\partial x^2}\right] dx \equiv 0$$

holds.

(ii) Now suppose that $\mu = \mu_0 + \epsilon\nu(x,T)$, where $\epsilon \ll 1$ and $T = \epsilon t$. Seek a solution to $(*)$ in the form of a moving front, by writing

$$A = A_0(x - \hat{X}(T)) + \epsilon R(x,T)\,,$$

where $\epsilon R(x,T)$ is a small remainder term. By substituting into $(*)$, derive the equation governing the front velocity

$$-\frac{d\hat{X}}{dT}\int_{-\infty}^{\infty}\left(\frac{\partial A_0}{\partial x}\right)^2 dx = \int_{-\infty}^{\infty} \nu(x + \hat{X}, T)A_0(x)\frac{\partial A_0(x)}{\partial x}dxz\,.$$

Find $d\hat{X}/dT$ explicitly in terms of $\hat{X}$ and T when $\nu(x,T) = K\delta(x + cT)$, where δ is the usual δ-function. Find conditions relating c and K which permit steady moving fronts with $d\hat{X}/dT$ =const. Are these solutions stable? Explain what happens to the front when these conditions are not met.

4 Write an essay on modulational instabilities of travelling wave patterns that arise as Hopf bifurcations in an extended system. You should include:

Reduction of the problem to the complex Ginzburg-Landau equation near the bifurcation point;

Discussion of the role of distant boundaries, and the need to use periodic boundary conditions;

Determination of the stability properties of "flat" solutions (modes of optimum wavelength);

Use of symmetry principles to derive the Kuramoto-Sivashinsky (KS) equation for long-wavelength disturbances; and

Discussion of small- but finite-amplitude solutions of the KS equation with finite spatial period P to determine the stability of modulated solutions.