UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 2 June 2006 9 to 11

PAPER 34

STOCHASTIC NETWORKS

Attempt ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define an *open migration process*. Establish the form of the equilibrium distribution, giving conditions for its existence.

Prove that the reversed process obtained from a stationary open migration process is itself an open migration process.

Show that if the parameters of a stationary open migration process are such that there is no path by which an individual leaving colony k can later reach colony j either directly or indirectly, then the stream of individuals moving directly from colony j to colony k forms a Poisson process.

2 Describe briefly a mathematical model for a loss network, and obtain an expression for the probability that an arriving call for route r is blocked.

A network consists of three nodes, with each pair of nodes connected by a link. A call in progress between two nodes may be routed on the direct link between the nodes, or on the two link path through the third node. A call in progress can be rerouted if this will allow an additional arriving call to be accepted. Describing carefully the modelling assumptions you make, obtain an expression for the probability an arriving call is blocked, for each of three possible node pairs.

3 Define a *Wardrop equilibrium* for the flows in a congested network.

Show that if the delay $D_j(y_j)$ at link j is a continuous, strictly increasing function of the throughput, y_j, of link j then a Wardrop equilibrium exists and solves an optimization problem of the form

$$\begin{aligned} \text{minimize} \quad & \sum_{j \in J} \int_0^{y_j} D_j(u)\, du \\ \text{over} \quad & x \geqslant 0,\ y \\ \text{subject to} \quad & Hx = f,\ Ax = y\ , \end{aligned}$$

where $f = (f_s, s \in S)$ and f_s is the (fixed) aggregate flow between source-sink pair s. What are the matrices A and H? In what sense is the equilibrium unique?

Suppose now that the aggregate flow between source-sink pair s is not fixed, but is a continuous, strictly decreasing function $B_s(\lambda_s)$, where λ_s is the minimal delay over all routes serving the source-sink pair s, for each $s \in S$. For the extended model, show that an equilibrium exists and solves the optimization problem

$$\begin{aligned} \text{minimize} \quad & \sum_{j \in J} \int_0^{y_j} D_j(u)\, du - G(f) \\ \text{over} \quad & x \geqslant 0,\ y, f \\ \text{subject to} \quad & Hx = f,\ Ax = y, \end{aligned}$$

for a suitable choice of the function $G(f)$.

4 Let J be a set of resources, and R a set of routes, where a route $r \in R$ identifies a subset of J. Let C_j be the capacity of resource j, and suppose the number of flows in progress on each route is given by the vector $n = (n_r, r \in R)$. A rate allocation $x = (x_r, r \in R)$ is *feasible* if

$$x_r \geqslant 0, \ r \in R, \text{ and } \sum_{r:j\in r} n_r x_r \leqslant C_j \quad j \in J.$$

A rate allocation is *proportionally fair* if it is feasible and if, for any other feasible rate allocation $y = (y_r, r \in R)$,

$$\sum_{r\in R} n_r \left(\frac{y_r - x_r}{x_r} \right) \leqslant 0.$$

Show that a proportionally fair rate allocation solves the optimization problem

$$\begin{aligned} \text{maximize} \quad & \sum_r n_r \log x_r \\ \text{over} \quad & x_r \geqslant 0, \quad r \in R \\ \text{subject to} \quad & \sum_{r:j\in r} n_r x_r \leqslant C_j \quad j \in J. \end{aligned}$$

Consider a network with resources $J = \{1,2,3,4\}$, each of unit capacity, and routes $R = \{\{1,2\},\{2,3\},\{3,4\},\{4,1\}\}$. Given $n = (n_r, r \in R)$, find the rate x_r of each flow on route r, for each $r \in R$, under a proportionally fair rate allocation. Show, in particular, that if $n_{\{1,2\}} > 0$ then

$$x_{\{1,2\}} n_{\{1,2\}} = \frac{n_{\{1,2\}} + n_{\{3,4\}}}{n_{\{1,2\}} + n_{\{2,3\}} + n_{\{3,4\}} + n_{\{4,1\}}}.$$

Suppose now that flows describe the transfer of documents through a network, that new flows originate as independent Poisson processes of rates $\nu_r, r \in R$, and that document sizes are independent and exponentially distributed with parameter μ_r for each route $r \in R$. Determine the transition intensities of the resulting Markov process $n = (n_r, r \in R)$. Show that the stationary distribution of the Markov process $n = (n_r, r \in R)$ takes the form

$$\pi(n) = B^{-1} \binom{n_{\{1,2\}} + n_{\{2,3\}} + n_{\{3,4\}} + n_{\{4,1\}}}{n_{\{1,2\}} + n_{\{3,4\}}} \prod_{r\in R} \left(\frac{\nu_r}{\mu_r} \right)^{n_r}.$$

END OF PAPER