UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 4 June 2003 9 to 11

PAPER 34

STOCHASTIC NETWORKS

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A telephone banking facility has N incoming lines and a single (human) operator. Calls to the facility are initiated as a Poisson process of rate ν, but calls initiated when all N lines are in use are lost. A call finding a free line has then to wait for the operator to answer. The operator deals with waiting calls one at a time, and takes an exponentially distributed length of time with parameter λ to check the caller's identity, after which the call is passed to an automated handling system for the caller to transact banking business, and the operator is freed to deal with another caller. The automated handling system is able to serve up to N callers simultaneously, and the time it takes to serve a call is exponentially distributed time with parameter μ. All these lengths of time are independent of each other and of the Poisson arrival process.

Model the facility as a closed migration process, and show that in equilibrium the proportion of calls lost is

$$H(N)\left(\sum_{n=0}^{N} H(n)\right)^{-1}$$

where

$$H(n) = \left(\frac{\nu}{\lambda}\right)^{n} \sum_{i=0}^{n} \left(\frac{\lambda}{\mu}\right)^{i} \frac{1}{i!}.$$

Develop an expression for the proportion of calls lost when the single operator is replaced by two operators able to deal with incoming calls.

2 Write an essay on mathematical models of loss networks. Your essay should cover the following topics, but need not be restricted to them.

(i) The stationary distribution for a loss network operating under fixed routing.

(ii) The Erlang fixed point approximation for a loss network, and its uniqueness when routing is fixed.

(iii) An example of a loss network with alternative routing where the Erlang fixed point approximation is not unique.

3 Define a *Wardrop equilibrium* for the flows in a congested network. Show that if the delay $D_j(\rho_j)$ at a link j is a continuous increasing function of the throughput ρ_j of link j, then a Wardrop equilibrium exists. Show that a Wardrop equilibrium solves an optimization problem of the form

$$\begin{aligned} \text{minimize} \quad & \sum_{j \in J} \int_0^{\rho_j} D_j(z)dz \\ \text{over} \quad & \nu, \rho \geqslant 0 \\ \text{subject to} \quad & H\nu = b, \quad A\nu = \rho. \end{aligned}$$

What are the matrices H and A? If, for each $j \in J$, the function $D_j(\rho_j)$ is strictly increasing, show that the equilibrium value of ρ is unique. Give an example to show that the equilibrium value of ν may not be unique.

Describe briefly an example of *Braess' paradox*, and interpret your example within the above optimization framework.

4 Let J be a set of resources, and R a set of routes, where a route $r \in R$ identifies a subset of J. Let C_j be the capacity of resource j, and suppose the number of flows in progress on each route is given by the vector $n = (n_r, r \in R)$. Define a proportionally fair rate allocation.

Consider a network with resources $J = \{1, 2, 3, 4\}$, each of unit capacity, and routes $R = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\}$. Given $n = (n_r, r \in R)$, find the rate x_r of each flow on route r, for each $r \in R$, under a proportionally fair rate allocation. Show, in particular, that if $n_{\{1,2\}} > 0$ then

$$x_{\{1,2\}} n_{\{1,2\}} = \frac{n_{\{1,2\}} + n_{\{3,4\}}}{n_{\{1,2\}} + n_{\{2,3\}} + n_{\{3,4\}} + n_{\{4,1\}}} .$$

Suppose now that flows describe the transfer of documents through a network, that new flows originate as independent Poisson processes of rates ν_r, $r \in R$, and that document sizes are independent and exponentially distributed with parameter μ_r for each route $r \in R$. Determine the transition intensities of the resulting Markov process $n = (n_r, r \in R)$. Show that the stationary distribution of the Markov process $n = (n_r, r \in R)$ takes the form

$$\pi(n) = B^{-1} \binom{n_{\{1,2\}} + n_{\{2,3\}} + n_{\{3,4\}} + n_{\{4,1\}}}{n_{\{1,2\}} + n_{\{3,4\}}} \prod_{r \in R} \left(\frac{\nu_r}{\mu_r} \right)^{n_r} .$$