UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 7 June 2002 9 to 11

PAPER 51

STOCHASTIC MODELS OF TRANSPORT AND MIXING

*Attempt **TWO** questions*

*There are **three** questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider the advection-diffusion problem

$$\chi_t + U(t)\sin my\ \chi_x = \kappa(\chi_{xx} + \chi_{yy})$$

with $\chi(x,y,t)$ the concentration and κ and m constants. The boundary conditions on χ are that χ is periodic in the y-direction, with $\chi(x,y,t) = \chi(x, y+2\pi/m, t)$ and that $\chi \to 0$ as $x \to \pm\infty$. $U(t)$ is a periodic function.

Use **either** the method of moments, considering $\int_{-\infty}^{\infty} \chi dx$, $\int_{-\infty}^{\infty} x\chi dx$, etc, **or** the homogenisation (multiple scales) technique, assuming that $\chi = \chi(\epsilon x, x, y, \epsilon^2 t, t)$ with ϵ small, to show that at large times $\bar{\chi}$, where $\bar{(.)}$ denotes average in y, satisfies a diffusion equation in x and t with diffusivity κ_e where

$$\kappa_e = \kappa + \tfrac{1}{2}\langle U(t)g(t)\rangle = \kappa + \frac{\kappa m^2}{2}\langle g(t)^2\rangle$$

where $\langle . \rangle$ denotes average over one time period and g is the time-periodic solution of $\dot{g} + \kappa m^2 g = U(t)$. Justify carefully the arguments that you use in either case.

Deduce κ_e when $U(t) = U_0$ (constant).

Now consider the case $U(t) = \sum_{n=-\infty}^{\infty} U_1 \delta(t - nT)$ where $\delta(.)$ is the Dirac delta function and U_1 is a constant. (The sum is over integer values of n.) Show that κ_e is now given by

$$\kappa_e = \kappa + \frac{U_1^2}{4T}\frac{(1 - e^{-2\kappa m^2 T})}{(1 - e^{-\kappa m^2 T})^2}.$$

(You should use the second expression for κ_e given above.)

Consider the limits $\kappa m^2 T \gg 1$ and $\kappa m^2 T \ll 1$ and carefully interpret the forms for κ_e that you find in those limits.

2 The concentration $\theta(\mathbf{x}, t)$ of an advected scalar satisfies

$$\theta_t + \mathbf{u}.\nabla\theta = 0,$$

where $\mathbf{u}(\mathbf{x}, t)$ is the velocity field, $\mathbf{x}$ represents position and t time. Show that $\nabla\theta$ satisfies the equation

$$\nabla\theta_t + (\mathbf{u}.\nabla)\nabla\theta = -(\nabla\mathbf{u})^T.\nabla\theta$$

where the ijth component of the tensor $\nabla\mathbf{u}$ is $\partial u_i/\partial x_j$. $\mathbf{k}(t)$ is defined as $\nabla\theta$ evaluated at a point $\mathbf{X}(t)$ following a fluid particle trajectory. Deduce that $\mathbf{k}(t)$ satisfies the equation

$$\frac{d\mathbf{k}}{dt} = -\sigma(t)^T.\mathbf{k}$$

where the tensor $\sigma(t)$ is $\nabla\mathbf{u}$ following the fluid particle, i.e. $\nabla\mathbf{u}$ evaluated at $\mathbf{X}(t)$.

Suppose that the vertical component of velocity $u_3 = 0$ and write down the corresponding equations for k_1, k_2 and k_3 (the components of $\mathbf{k}$) in terms of the non-zero components of $\sigma(t)$. For simplicity assume that $\sigma_{11} = -\sigma_{22} = a$, $\sigma_{12} = \sigma_{21} = b$, where a and b are constant in time. Integrate the ordinary differential equations for k_1 and k_2 to show that at large times $k_1 \simeq k_0 \cos\phi e^{ct}$ and $k_2 \simeq k_0 \sin\phi e^{ct}$, where $c = \sqrt{a^2 + b^2}$, for suitable chosen constants k_0 and ϕ.

Now consider the following two cases:

Case I: σ_{13} and σ_{23} are constant in time.

Show that k_3 increases exponentially at large times and that

$$\alpha = \frac{|k_3|}{(k_1^2 + k_2^2)^{1/2}} \rightarrow \frac{|\sigma_{13}\cos\phi + \sigma_{23}\sin\phi|}{\sqrt{a^2 + b^2}}$$

as $t \to \infty$. What is the physical interpretation of α?

Now assume that a and b are independent Gaussian random variables with zero mean and variance Γ^2 and that $\sigma_{13}\cos\phi + \sigma_{23}\sin\phi$ is another Gaussian random variable with zero mean and variance Λ^2.

Considering the long-time limit, evaluate, as a function of A, the probability that $\alpha \leq A$ and deduce that the probability density function for α is

$$p_\alpha(A) = \frac{\Lambda^2\Gamma}{(\Gamma^2 A^2 + \Lambda^2)^{3/2}}.$$

If a, b, σ_{13} and σ_{23} varied randomly in time on some time scale τ_σ and had statistics as assumed above, what condition on τ_σ would be required for above form for the probability density function for α to be a useful approximation?

Case II: σ_{13} and σ_{23} vary randomly and rapidly in time, so that $\sigma_{13}dt = gdW^{(1)}$ and $\sigma_{23}dt = gdW^{(2)}$, where g is a constant and $W^{(1)}$ and $W^{(2)}$ are independent Wiener processes. Deduce that at large times k_3 satisfies the equation

$$dk_3 = -k_0 g e^{ct}\cos\phi dW^{(1)} \quad - k_0 g e^{ct}\sin\phi dW^{(2)} = -k_0 g e^{ct} dW$$

($c = \sqrt{a^2 + b^2}$ as above, and may be treated as a constant), with W another Wiener process. Derive a stochastic differential equation for $\beta = k_3/\sqrt{k_1^2 + k_2^2} = k_3 e^{-ct}/k_0$.

Write down the corresponding Fokker-Planck equation for the probability density function of β and integrate to show that the steady-state form of the probability density function (given c) is

$$p_\beta(B;c) = \left(\frac{c}{\pi g^2}\right)^{1/2} \exp(-cB^2/g^2).$$

(Note that β may take either positive or negative values.)

3 Consider the evolution of an advected scalar in a two-dimensional turbulent incompressible flow, on the assumption that the scalar diffusivity is much smaller than the momentum diffusivity, so that the minimum length scales in the scalar field are much smaller than the minimum length scales in the velocity field. Justify considering the system

$$\chi_t + (C(t).\mathbf{x}).\nabla\chi = \kappa\nabla^2\chi \qquad (*)$$

where C is a tensor that is a random function of time and then averaging over all realisations of C.

From $(*)$ derive equations for $\int \chi dA$ and for the 2nd moment tensor $J_{ij} = \int x_i x_j \chi dA$, where the integrals are taken over the infinite two-dimensional domain on which $(*)$ applies and it may be assumed that χ decays rapidly for large $|\mathbf{x}|$.

Consider solutions to $(*)$ of the form

$$\chi(\mathbf{x},t) = f(t)\exp(-\mathbf{x}^T B(t)\mathbf{x})$$

where $B(t)$ is a symmetric matrix. Derive the relation between $f(t)$ and $B(t)$ and $J(t)$ (i.e. the matrix of components of the 2nd moment tensor, evaluated for this solution).

Discuss the statistical behaviour of the eigenvalues of J, $e^{2\rho_1}$ and $e^{2\rho_2}$ say, at large times. Assume that in the initial condition the length scales are large enough such that diffusion may be neglected and make it clear how diffusion affects the behaviour at large times. (You need not give detailed mathematical derivations e.g. of probability density functions, but results should be clearly stated.)

Describe briefly how these results may be used to describe the decay of a scalar in a turbulent flow. Show that the μth moment $\langle|\chi(\mathbf{0},t)|^\mu\rangle$ $(\mu > 0)$, where the expectation $\langle . \rangle$ is over all realisations of the flow, decays as $e^{-\gamma_\mu t}$, where each γ_μ is a constant, and give equations for γ_μ. Show that γ_μ may be independent of μ if μ is larger than some critical value μ_* and show also that, for $\mu < \mu_*$, γ_μ/μ is a decreasing function of μ.