MATHEMATICAL TRIPOS Part III

Monday 9 June 2008 1.30 to 3.30

# PAPER 36

# STOCHASTIC LOEWNER EVOLUTIONS

*Attempt no more than* **THREE** *questions.*

*There are* **FOUR** *questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** (a) Let $K$ be a compact $\mathbb{H}$-hull and set $H = \mathbb{H} \setminus K$. Show that there exists a unique conformal isomorphism $g_K : H \to \mathbb{H}$ such that $g_K(z) - z \to 0$ as $z \to \infty$.

(b) Suppose that $K_0$ is a compact $\mathbb{H}$-hull and $g_{K_0}(z) = z + z^{-1}$ for all $z \in \mathbb{H} \setminus K_0$. Identify $K_0$ and find $\operatorname{hcap}(K_0)$.

(c) Suppose that $|z| \leq 1$ for all $z \in K$. Show that, for all $x \in \mathbb{R}$ with $|x| > 1$,

$$1 - x^{-2} \leq g_K'(x) \leq 1.$$

[In (c), you may use without proof any result from the course.]

**2** (a) Let $(\gamma_t)_{t\geq 0}$ be an SLE($\kappa$), for some $\kappa \in [0, \infty)$. Explain the relation to $(\gamma_t)_{t\geq 0}$ of the associated Loewner flow $(g_t)_{t\geq 0}$ and transform $(\xi_t)_{t\geq 0}$.

(b) Fix $s \geq 0$ and define for $t \geq 0$

$$\bar{\gamma}_t = g_s(\gamma_{s+t}), \quad \tilde{\gamma}_t = \bar{\gamma}_t - \xi_s.$$

What is the Loewner transform of $(\bar{\gamma}_t)_{t\geq 0}$? What is the distribution of $(\tilde{\gamma}_t)_{t\geq 0}$? Justify your answers.

(c) Suppose now that $\kappa \in (0, 4]$. Show that, almost surely, $(\gamma_t)_{t\geq 0}$ is a simple curve. [You may assume without proof that, almost surely, $\operatorname{Im}(\gamma_t) > 0$ for all $t > 0$.]

**3** (a) Let $\gamma$ be an SLE(8/3). Let $U$ be a simply connected domain in the upper half-plane $\mathbb{H}$, which is a neighbourhood of both 0 and $\infty$. Denote by $\Phi$ the unique conformal isomorphism $U \to \mathbb{H}$ such that $\Phi(z) - z \to 0$ as $z \to \infty$. Set $K_t = \{\gamma_s : 0 < s \leq t\}$ and $T = \inf\{t \geq 0 : \gamma_t \notin U\}$. Define, for $t < T$,

$$K_t^* = \{\Phi(\gamma_s) : 0 < s \leq t\}, \quad \Phi_t = g_{K_t^*} \circ \Phi \circ g_{K_t}^{-1},$$

where, for $K$ a compact $\mathbb{H}$-hull, $g_K : (\mathbb{H} \setminus K) \to \mathbb{H}$ is the unique conformal isomorphism such that $g_K(z) - z \to 0$ as $z \to \infty$. Set

$$\Sigma_t = \Phi_t'(\xi_t),$$

where $\xi$ is the Loewner transform of $\gamma$. Show that a suitably chosen function of the process $\Sigma$ is a local martingale.

(b) Hence, show that

$$\mathbb{P}(\gamma_t \in U \text{ for all } t \geq 0) = \Phi'(0)^{5/8}.$$

[You may assume without proof any standard identities of the classical Loewner theory, or for the Brownian excursion. You may also assume that $\Sigma_t \to 1_{\{T=\infty\}}$ as $t \uparrow T$, almost surely.]

**4** (a) Let $\mu$ be a scale-invariant probability measure on chords in the upper half-plane from 0 to $\infty$. What does it mean to say that $\mu$ has the locality property?

(b) Let $\gamma$ be an SLE(6) and let $\Phi : N \to N^*$ be a conformal isomorphism of one neighbourhood of 0 in $\mathbb{H}$ to another. Assume that $\Phi(0) = 0$ and that $\Phi(\bar{N} \cap \mathbb{R}) = \overline{N^*} \cap \mathbb{R}$. Set $T = \inf\{t \geq 0 : \gamma_t \notin N\}$ and define, for $t < T$,

$$\gamma_t^* = \Phi(\gamma_t).$$

Write $(\xi_t^*)_{t<T}$ for the Loewner transform of $(\gamma_t^*)_{t<T}$. Show that $(\xi_t^*)_{t<T}$ is a local martingale.

(c) Deduce that the law of $[\gamma]$ has the locality property.

## END OF PAPER