UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 8 June 2007 1.30 to 3.30

PAPER 39

STOCHASTIC LOEWNER EVOLUTIONS

Attempt ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $(K_t)_{t \geqslant 0}$ be a strictly increasing family of *compact* $\mathbb{H}$-*hulls* with $hcap(K_t) = 2t$ for all t and with the *local growth property*, having *Loewner transform* $(\xi_t)_{t \geqslant 0}$. Explain all the italicized terms in the preceding sentence and state how one can obtain $(\xi_t)_{t \geqslant 0}$ from $(K_t)_{t \geqslant 0}$. Discuss briefly also how to reconstruct $(K_t)_{t \geqslant 0}$ from $(\xi_t)_{t \geqslant 0}$.

Let $\kappa \in [0, \infty)$. What is meant by saying that a continuous process $(\gamma_t)_{t \geqslant 0}$ is (chordal, half-plane) SLE(κ)?

Show carefully that SLE(κ) is scale-invariant.

Explain how it is possible to define in a consistent way SLE(κ) in any simply connected Jordan domain D from one given boundary point z_0 to another z_1. (You may assume that $|\gamma_t| \to \infty$ as $t \to \infty$ almost surely.)

2 Let γ be an SLE(κ), with $\kappa \in (0, 4)$. Show that, almost surely, γ is a simple curve and $|\gamma_t| \to \infty$ as $t \to \infty$. You may use any facts about Bessel processes you wish, without proof, provided that these are clearly stated.

3 Fix $a \in (0, 1/2)$. For $x, y \in (0, \infty)$, define processes X and Y by the stochastic differential equations

$$dX_t = dB_t + \frac{a}{X_t}, \quad dY_t = -dB_t + \frac{a}{Y_t}, \quad X_0 = x, \quad Y_0 = y,$$

where B is a Brownian motion, and we consider X as defined up to the first time ζ that it hits 0, and similarly Y as defined up to the first time τ that it hits 0. Show that $\zeta < \infty$, almost surely.

Show moreover that

$$\mathbb{P}(\zeta < \tau) = c \int_0^{y/(x+y)} \frac{du}{u^{2-4a}(1-u)^{2a}},$$

for some constant c independent of x and y.

Discuss briefly how this probability can be interpreted, for a suitable value of a, which you should specify, as a crossing probability for the continuum limit of critical planar percolation.

4 Explain what is meant by a *filling* of $D = (U, z_0, z_1)$, where U is a simply connected complex domain, and z_0, z_1 are two distinct points of the conformal boundary of U.

Suppose that $(\mu_D : D \in \mathcal{D})$ is a conformally invariant family of probability measures, indexed by the set $\mathcal{D}$ of all such D, where μ_D is a measure on fillings in D. What does it mean to say that $(\mu_D : D \in \mathcal{D})$ has the restriction property? Express this property in terms of the single measure μ corresponding to $D = (\mathbb{H}, 0, \infty)$.

Let γ be an $SLE(8/3)$ and let E be a Brownian excursion in $\mathbb{H}$, from 0 to ∞. Thus $E = (B, |W|)$ with B and W independent Brownian motions, starting from 0, in $\mathbb{R}$ and $\mathbb{R}^3$ respectively. For D a filling of $(\mathbb{H}, 0, \infty)$, it is known that

$$\mathbb{P}(\gamma_t \in D \text{ for all } t \geqslant 0) = (\Phi'_D(0))^{5/8}, \quad \mathbb{P}(E_t \in D \text{ for all } t \geqslant 0) = \Phi'_D(0),$$

where Φ_D is the conformal isomorphism $D \to (\mathbb{H}, 0, \infty)$ with $\Phi'_D(\infty) = 1$. [*You are not expected to prove these facts in this question.*] Deduce that the fillings $\bar{\gamma}$ and $\bar{E}$, generated by γ and E respectively, both have the restriction property.

Show further that the law of the filling $\bar{\gamma}^{\otimes 8}$ generated by eight independent copies of γ coincides with the law of the filling $\bar{E}^{\otimes 5}$ generated by five independent copies of E.

END OF PAPER