UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 4 June 2007 1.30 to 4.30

PAPER 36

STOCHASTIC CALCULUS AND ITS APPLICATIONS

*Attempt **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Define what it means for a process to be a local martingale.
Let X be a local martingale. State a necessary and sufficient condition for X to be a true martingale, and define all the terms used.

(b) State and prove Lévy's characterization of Brownian motion in $\mathbb{R}^d$, for $d \geqslant 1$.

Let B be a standard Brownian motion in $\mathbb{R}^2$ and consider $X_t = \log |B_t|$ for $t \geqslant 1$.

(c) Show that X is a local martingale.

(d) Show that $\mathbb{E}\big(|X_t|^p\big) < \infty$ for any $p \in (0, \infty)$ but $\lim_{t \to \infty} \mathbb{E}(X_t) = \infty$, so X is not a martingale.

2 (a) Define what is meant by the previsible σ-algebra and a previsible process.
For $M \in \mathcal{M}_c^2$ define the space $L^2(M)$.

(b) State the Itô isometry and define the Itô integral $H \cdot M$ for $H \in L^2(M)$ and $M \in \mathcal{M}_c^2$.
Explain how to extend the integral to locally bounded, previsible H and $M \in \mathcal{M}_{c,loc}$.

(c) State the theorem on existence and uniqueness of the quadratic variation.

(d) Show that $[H \cdot M] = H^2 \cdot M$.

(e) Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P})$ be a filtered probability space and $\mathbb{Q}$ be a measure such that $\mathbb{Q} \ll \mathbb{P}$. Suppose that X is a semimartingale with respect to $\mathbb{P}$ and $\mathbb{Q}$, i.e.

$$X = M + A = N + B \qquad (X_0 = 0) ,$$

where M, N are the local martingale parts and A, B the bounded variation parts under $\mathbb{P}$ and $\mathbb{Q}$, respectively.
Starting with simple integrands or otherwise, show that the stochastic integral $(H \cdot X)_t$ is the same under $\mathbb{P}$ and $\mathbb{Q}$ for all locally bounded previsible H.
You may use any results you wish from the lectures and example sheets, without proof.

3 (a) State the integration by parts formula.
State Itô's formula for a d-dimensional continuous semimartingale.
Prove Itô's formula in one dimension for $f(X_t)$, where f is a polynomial.
You may use any results you wish concerning stochastic integration, without proof.

Let B be a standard Brownian motion in $\mathbb{R}$ and let $\beta_k(t) = \mathbb{E}(B_t^k)$ for $k = 0, 1, \ldots$.

(b) Show that $\beta_k(t) = \frac{1}{2}k(k-1)\int_0^t \beta_{k-2}(s)\,ds$ for all $k \geqslant 2$.
Any localization argument in your proof should be carried out in detail.

(c) Show that $\beta_{2k+1}(t) = 0$ and $\beta_{2k}(t) = \dfrac{(2k)!\,t^k}{2^k k!}$ for all $k \geqslant 0$.

(d) Is $B_t^k - b_k(t)$ a martingale for all $k \geqslant 1$? Justify your answer.

4 Consider the 1-dimensional stochastic differential equation

$$dX_t = \sigma(X_t)\,dB_t + b(X_t)\,dt \qquad (SDE)$$

where B is a standard Brownian motion in $\mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$, $b : \mathbb{R} \to \mathbb{R}$ are bounded and Lipschitz.

(a) Define the notions of weak solution, strong solution, uniqueness in law and pathwise uniqueness.
You do not have to write what is meant by a solution to (SDE).

Let X be a solution of (SDE) starting at $X_0 = x_0$.

(b) Show that $\dfrac{d}{dt}\mathbb{E}(X_t)\big|_{t=0} = b(x_0)$ and $\dfrac{d}{dt}\mathrm{Var}(X_t)\big|_{t=0} = \sigma^2(x_0)$.

Let B be a standard Brownian motion in $\mathbb{R}$ and consider

$$X_t = \cos B_t \quad \text{and} \quad Y_t = \sin B_t \ .$$

(c) Find the stochastic differential equation for the process (X, Y) and write it in its standard form, i.e. identify the functions $b(x, y)$ and $\sigma(x, y)$.

(d) Does the equation you found in (c) have a solution for each initial value (x_0, y_0)? Justify your answer by quoting the relevant result. Comment also on pathwise uniqueness and uniqueness in law.

(e) For every starting point (x_0, y_0), determine a strong solution of the equation you found in (c).

5 Let M be a continuous local martingale with quadratic variation $[M]$, such that

$$[M]_t = \int_0^t Q_s ds \to \infty \text{ as } t \to \infty,$$

for some continuous adapted process Q. Let A be an adapted process such that $|A_t| \leqslant CQ_t$ for all $t \geqslant 0$, for some constant $C < \infty$. Fix $Y_0 \in (-1, 1)$, set

$$Y_t = Y_0 + M_t + \int_0^t A_s ds$$

and set $T = \inf\{t \geqslant 0 : |Y_t| \geqslant 1\}$. Consider first the case where $Q_t \equiv 1$. Show that there is a constant $\varepsilon > 0$, depending only on C, such that $\mathbb{P}(T > 1) \leqslant 1 - \varepsilon$ and deduce that $T < \infty$ almost surely.

Extend this conclusion to the case of general Q.

Suppose that the stochastic differential equation

$$dX_t = \sum_{k=1}^{m} \sigma_k(X_t) dB_t^k + b(X_t)dt,$$

has, for each $x \in \mathbb{R}^d$, a solution $(X_t^x)_{t \geqslant 0}$ starting from x. Here $\sigma_1, \ldots, \sigma_m, b$ are bounded measurable vector fields on $\mathbb{R}^d$ and $B = (B^1, \ldots, B^m)$ is a Brownian motion in $\mathbb{R}^m$. Show that, for a certain differential operator L, which you should make explicit, $(X_t^x)_{t \geqslant 0}$ is an L-diffusion.

Assume that there is a constant $\lambda > 0$ such that, for all $x, \xi \in \mathbb{R}^d$,

$$\sum_{k=1}^{m} \langle \xi, \sigma_k(x) \rangle^2 \geqslant \lambda |\xi|^2.$$

Show that, for all $x \in \mathbb{R}^d$ and all $\xi \in \mathbb{R}^d \setminus \{0\}$,

$$\sup_{t \geqslant 0} |\langle \xi, X_t^x \rangle| = \infty, \text{ a.s.}$$

Let $u, v \in C_b^2(\mathbb{R}^d)$ and let D be bounded domain in $\mathbb{R}^d$, having a smooth boundary ∂D. Suppose that $u = v$ on ∂D. Show, in the case that $Lu = Lv = 0$ in D, that $u = v$ in D.

6 Explain what is meant by a Markov jump process $(X_t)_{t \geqslant 0}$, adapted to a filtration $(\mathcal{F}_t)_{t \geqslant 0}$.

State and prove a form of Kurtz' theorem on the approximation of Markov jump processes by the solutions of differential equations. You may use any sort of martingale inequalities you wish, without proof, provided that you make a clear statement.

Illustrate your result by the example of a population of cells, in which each cell divides to produce two identical cells at an exponential rate $1 - (\xi_t/N)$, where ξ_t is the total number of cells present at time t and N is a given large positive integer. You should assume an initial population of size pN for some $p \in (0, 1)$.

END OF PAPER