UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 9 June, 2006 9 to 12

PAPER 35

STOCHASTIC CALCULUS AND APPLICATIONS

*Attempt **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Define what it means for a process to be a local martingale.

(b) Suppose that X is a continuous local martingale of finite variation, with $X_0 = 0$. Show that $X \equiv 0$ almost surely. (You may assume that the total variation process of X is adapted and continuous.)

(c) Deduce that if Y is another continuous local martingale then there can be at most one continuous adapted increasing process A such that $Y^2 - A$ is again a continuous local martingale.

(d) Suppose now that H is a simple process, i.e.

$$H = \sum_{k=0}^{n-1} Z_k \mathbb{1}_{(t_k, t_{k+1}]},$$

where $n \in \mathbb{N}$, $0 = t_0 < t_1 < \ldots < t_n < \infty$ and Z_k is a bounded $\mathcal{F}_{t_k}$-measurable random variable for each k. Suppose that M is an L^2-bounded martingale with quadratic variation process $[M]$. Define the stochastic integral $H \cdot M$ and prove that

$$\mathbb{E}[(H \cdot M)_\infty^2] = \mathbb{E}[(H^2 \cdot [M])_\infty].$$

(You may assume that the Lebesgue-Stieltjes integral on the right-hand side is well-defined and that $H \cdot M$ and $M^2 - [M]$ are martingales.)

2 Suppose that B is a standard Brownian motion and that H is a locally bounded previsible process such that $\int_0^t H_s^2 ds$ is strictly increasing in t, $\int_0^t H_s^2 ds < \infty$ for all $t > 0$ and $\int_0^\infty H_s^2 ds = \infty$.

(a) Set $T = \inf\{t \geq 0 : \int_0^t H_s^2 ds > \sigma^2\}$, where $\sigma \neq 0$. Prove that

$$\int_0^T H_s dB_s \sim \mathrm{N}(0, \sigma^2).$$

(b) State the Dubins-Schwarz theorem for a local martingale M.

(c) Using part (a), or otherwise, prove the Dubins-Schwarz theorem in the special case where $M_t = \int_0^t H_s dB_s$.

3 (a) Suppose that W is a Brownian motion and set

$$A_t = \int_0^t \operatorname{sgn}(W_s) dW_s,$$

where

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Show that A is another Brownian motion and that if $V_t = W_t^2$ then

$$dV_t = 2\sqrt{V_t} dA_t + dt$$

(here, $\sqrt{x}$ is the non-negative square root of $x \in [0, \infty)$).

(b) Let $B^{(1)}$ and $B^{(2)}$ be independent Brownian motions and suppose that $\alpha \geq 0$ and $\beta \geq 0$ are constants. Let X satisfy

$$dX_t = 2\sqrt{X_t} dB_t^{(1)} + \alpha dt, \quad X_0 = x \geq 0$$

and let Y satisfy

$$dY_t = 2\sqrt{Y_t} dB_t^{(2)} + \beta dt, \quad Y_0 = y \geq 0.$$

Show that $Z = X + Y$ satisfies

$$dZ_t = 2\sqrt{Z_t} dB_t + \gamma dt, \qquad (\star)$$

where B is another Brownian motion and γ is a constant which you should determine.

(c) Suppose that Z is a solution to the stochastic differential equation $(\star)$ for some $\gamma \geq 2$, with $Z_0 = r^2$ and $r > 0$. Set $R_t = \sqrt{Z_t}$ and find a stochastic differential equation (which we will refer to as **(SDE)**) satisfied by R, at least until time $\zeta = \inf\{t \geq 0 : R_t = 0\}$.

(d) What does it mean for uniqueness in law to hold for a stochastic differential equation? Assume that uniqueness in law holds for **(SDE)** and suppose also that $\gamma \in \mathbb{Z}$ and $\gamma \geq 2$. Argue carefully that any solution to **(SDE)** must have the same distribution as the Euclidean norm of a γ-dimensional Brownian motion started from the sphere $\{x \in \mathbb{R}^\gamma : |x| = r\}$.

4 (a) Suppose that $\sigma : \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions. Prove that there is pathwise uniqueness for the stochastic differential equation

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt.$$

(You may use Gronwall's lemma without proof.)

(b) Henceforth, consider the special case

$$dX_t = dB_t + X_t dt, \quad X_0 = 0.$$

By means of an exponential integrating factor, find the (pathwise unique) solution.

(c) Let $T = \inf\{t \geq 0 : X_t = 1 \text{ or } X_t = -1\}$. Suppose that under the probability measure $\widetilde{\mathbb{P}}$, X is a Brownian motion. Using Girsanov's theorem, find a new probability measure $\mathbb{P}$, absolutely continuous with respect to $\widetilde{\mathbb{P}}$, such that B is a Brownian motion under $\mathbb{P}$, at least until time T.

(d) Show that

$$\mathbb{P}(T \leq t) \geq \exp\left(\frac{1}{2} - t\right) \widetilde{\mathbb{P}}(T \leq t).$$

(You may find it helpful to use Itô's formula to give an alternative expression for $\int_0^t X_s dX_s$.)

5 Consider, for $\varepsilon > 0$, the unique solution $u^\varepsilon \in C_b^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d)$ of the Cauchy problem

$$\begin{cases} \frac{\partial u^\varepsilon}{\partial t} = L^\varepsilon u^\varepsilon & \text{on } (0,\infty) \times \mathbb{R}^d, \\ u^\varepsilon(0,.) = f & \text{on } \mathbb{R}^d. \end{cases}$$

Here,

$$L^\varepsilon = \frac{\varepsilon^2}{2}\Delta + b(x).\nabla + c(x),$$

with b a Lipschitz vector field on $\mathbb{R}^d$, $c \in C_b(\mathbb{R}^d)$, and $f \in C_b^2(\mathbb{R}^d)$. Fix $x_0 \in \mathbb{R}^d$ and let $(x_t)_{t \geq 0}$ be the unique solution to the differential equation $\dot{x}_t = b(x_t)$ starting from x_0. Show that, for all $t \geq 0$, as $\varepsilon \downarrow 0$,

$$u^\varepsilon(t, x_0) \to f(x_t) \exp\left\{\int_0^t c(x_s)\, ds\right\}.$$

You may use any result from the course without proof, provided that you state it clearly.

6 State what is meant by a Markov jump process X with state-space $(E, \mathcal{E})$ and generator Q, adapted to a given filtration $(\mathcal{F}_t)_{t \geq 0}$.

Write μ for the jump measure of X on $(0, \infty) \times E$, given by

$$\mu = \sum_{X_t \neq X_{t-}} \delta_{(t, X_t)}.$$

State a general result which allows one to identify martingales associated with X in terms of μ.

Consider now the case where X is a birth process with rates $\lambda(1), \lambda(2), \ldots$. Thus X has state-space $\mathbb{N}$ and, for all $t \geq 0$ and $i \in \mathbb{N}$, at time t, conditional on $X_t = i$, X jumps to $i+1$ at rate $\lambda(i)$. Fix $\theta \in \mathbb{R}$ and set

$$M_t = \exp\left\{\theta X_t - (e^\theta - 1) \int_0^t \lambda(X_s) ds\right\}.$$

Show that, for any value of θ, M is a local martingale up to the explosion time ζ of X. Show further that, if the rates are uniformly bounded, then M is in fact a martingale.

END OF PAPER