UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 6 June 2003 9 to 12

PAPER 33

STOCHASTIC CALCULUS AND APPLICATIONS

Attempt **FOUR** *questions.*

There are **six** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Let M and A be continuous adapted processes. What does it mean to say that M is a local martingale? What does it mean to say that A has finite variation? Show that if $M = A$ then $M_t = M_0$ for all $t \geqslant 0$. You may assume without proof that the total variation process of A is continuous and adapted.

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function with continuous derivatives of first and second order. Set $X_t = f(M_t, A_t)$, $t \geqslant 0$. Show, stating clearly any standard theorems to which you appeal, that X may be expressed as the sum of a continuous local martingale M^f and a continuous adapted process A^f of finite variation, with $A_0^f = 0$.

(c) Consider the case where M is a Brownian motion and where $A_t = t$ for all $t \geqslant 0$. Determine for which functions f we have $A_t^f = 0$ for all $t \geqslant 0$ almost surely.

2 (a) Let M be a continuous local martingale and let H be a locally bounded previsible process. Show that the Itô integral $H \cdot M$ has quadratic variation process $H^2 \cdot [M]$, where $[M]$ is the quadratic variation process of M. Any localization argument you use should be set out in detail. Standard properties of the Itô isometry defining the Itô integral may be assumed without proof.

(b) State and prove Lévy's characterization of Brownian motion. The one-dimensional case will suffice.

(c) Which of the following processes, defined in terms of two independent Brownian motions B and W, are themselves Brownian motions?

(i) $X_t = \int_0^t sgn(B_s) dB_s$,

(ii) $Y_t = e^{W_t} \cos B_t$,

(iii) $Z_t = Y_{\tau_t}$, $\tau_t = \inf\{u \geqslant 0 : \int_0^u e^{2W_s} ds = t\}$.

Justify your answers.

3 Compute, for each $t > 0$, the distribution of X_t and Y_t, given by the stochastic differential equations

$$\begin{aligned} dX_t &= \sigma\, dB_t - \lambda X_t dt, \quad X_0 = 1, \\ dY_t &= dW_t + (2Y_t)^{-1} dt, \quad Y_0 = 1. \end{aligned}$$

Here B and W are independent Brownian motions; σ and λ are constants, with $\sigma > 0$. You may assume that the distribution of Y is determined by the distribution of W and its stochastic differential equation. You may express your answers in terms of the distributions of suitable functions of Gaussian random variables.

4 Consider the Cauchy problem for a function $u = u^\lambda \in C_b^2([0,\infty) \times \mathbb{R})$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} - \lambda\phi(x)u, & \text{on } [0,\infty) \times \mathbb{R}, \\ u(0,\cdot) = 1, & \text{on } \mathbb{R}. \end{cases}$$

Here λ is a positive constant and

$$\phi(x) = 0 \vee (-x) \wedge 1.$$

Show that if u is a solution then

$$u(t,x) = \mathbb{E}_x\big(\exp\{-\lambda \int_0^t \phi(B_s)ds\}\big)$$

where B is a Brownian motion starting from x.

Hence show that $u^\lambda(t,x) \to v(t,x)$ as $\lambda \to \infty$ for all $(t,x) \in (0,\infty) \times \mathbb{R}$, where

$$v(t,x) = \begin{cases} 1 - 2\mathbb{P}_x(B_t \leqslant 0), & x \geqslant 0, \\ 0\,, & x < 0. \end{cases}$$

5 Let U and V be independent random variables, each uniformly distributed on $(0,1)$. Set

$$X_t = U 1_{\{V \geqslant e^{-t}\}},\ \mathcal{F}_t = \sigma(X_s : s \leqslant t), \quad t \geqslant 0.$$

(a) Show from first principles that, for any Borel subset B of $(0,1)$, the following process is a martingale:

$$M_t = 1_{\{T \leqslant t, X_t \in B\}} - \mathbb{P}(U \in B)(T \wedge t), \quad t \geqslant 0,$$

for some stopping time $T > 0$ be to determined.

(b) Show that $(X_t)_{t \geqslant 0}$ is a Markov jump process with state-space $[0,1)$ and identify its Lévy kernel. Hence, stating clearly any general theorems to which you appeal, find, for each real θ a previsible process $(Y_t)_{t \geqslant 0}$ such that the following process is a martingale:

$$Z_t = \exp\{\theta X_t - Y_t\}, \quad t \geqslant 0.$$

6 Two rival gangs engage in a gunfight. The number in each gang still able to fight decreases at a rate proportional to the number in the opposing gang, with the same constant of proportionality on both sides. Assume that, initially, gang A has N members and gang B has αN members where $\alpha \in (0,1)$. Model the fight as a Markov jump process.

Let X^N denote the proportion of gang A left standing after the fight. Show that, for all $\delta > 0$

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}(|X^N - \sqrt{1-\alpha^2}| > \delta) < 0\ .$$