UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 5 June 2002 9 to 12

PAPER 31

STOCHASTIC CALCULUS AND APPLICATIONS

Attempt ***FOUR*** *questions*

There are ***six*** *questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider the stochastic differential equation in $\mathbb{R}$

$$dX_t = \sigma(X_t)dB_t, \quad X_0 = 0$$

where B is a Brownian motion in $\mathbb{R}$. Assume that X is a solution and that σ is a bounded measurable function on $\mathbb{R}$. Write $v = \sigma^2$ and set

$$Z_t = X_t^2 - \int_0^t v(X_s)ds.$$

Show that both X and Z are martingales.

Let now X be a pure jump Markov process in $\mathbb{R}$, starting from 0 and having Lévy kernel K. Assume that

$$\int_{\mathbb{R}} K(x,dy) = \lambda(x), \quad \int_{\mathbb{R}} yK(x,dy) = 0, \quad \int_{\mathbb{R}} y^2 K(x,dy) = v(x)$$

where λ and v are bounded functions on $\mathbb{R}$. Set

$$Z_t = X_t^2 - \int_0^t v(X_s)ds.$$

Show that both X and Z are martingales.

2 Let B be a Brownian motion in $\mathbb{R}^2$ with $|B_0| = 1$. Set $M_t = \log|B_t|$ and, for $r \in (0,1)$ and $R \in (1,\infty)$, set

$$T = T(R,r) = \inf\{t \geq 0 : |B_t| \in \{r, R\}\}.$$

Show that M is a local martingale, at least up to the first time B hits 0. Hence show that $\mathbb{E}(M_T) = 0$ and deduce that $B_t \neq 0$ for all $t \geqslant 0$ almost surely. Show further that M is not a martingale. [*You may wish to use, with justification, following inequality:*

$$M^2_{T(2,0)\wedge t} \leqslant (\log 2)^2 + M_t^2 1_{|B_t| \leqslant 1/2}.]$$

3 Let M be a continuous L^2-bounded martingale with quadratic variation $[M]$. Define the integral $H \cdot M$ of a simple process H with respect to M and show that

$$\mathbb{E}[(H \cdot M)_\infty^2] = \mathbb{E}[(H^2 \cdot [M])_\infty].$$

Let now H be a previsible process such that

$$\mathbb{E} \int_0^1 H_s^2 ds < \infty.$$

Show that there exists a sequence of simple processes H^n such that

$$\mathbb{E} \int_0^1 (H_s^n - H_s)^2 ds \to 0.$$

Let B be a Brownian motion in $\mathbb{R}$. Explain how to define the Itô integral $\int_0^1 H_s dB_s$ of H with respect to B.

4 Let B be a Brownian motion in $\mathbb{R}$, starting from 0. Show that, for all $\delta > 0$ and $t \geqslant 0$,

$$\mathbb{P}\left(\sup_{s \leqslant t} |B_s| > \delta\right) \leqslant 2e^{-\delta^2/(2t)}.$$

Let $b : \mathbb{R}^d \to \mathbb{R}^d$ be a Lipschitz function and let $x_0 \in \mathbb{R}^d$. Consider, for each $\varepsilon > 0$, the diffusion process X^ε in $\mathbb{R}^d$, starting from x_0 and having generator

$$L^\varepsilon = \frac{1}{2}\varepsilon^2 \Delta + b(x).\nabla.$$

Show that, for all $\delta > 0$ and $t \geqslant 0$,

$$\limsup_{\varepsilon \downarrow 0} \varepsilon^2 \log \mathbb{P}\left(\sup_{s \leqslant t} |X_s^\varepsilon - x_s| > \delta\right) < 0$$

where $(x_t)_{t \geqslant 0}$ satisfies $\dot{x}_t = b(x_t)$.

5 Let X and Y be independent Brownian motions in $\mathbb{R}$ and set $Z_t = X_t + iY_t$. Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function and set $M_t = U_t + iV_t = f(Z_t)$. Show that U and V are local martingales and compute their quadratic variations $[U]$ and $[V]$ and covariation $[U, V]$.

Set $D = \{x + iy : x, y > 0\}$. Fix $a > 0$ and take Z as above, where now $X_0 = Y_0 = a$. Set

$$T = \inf\{t \geqslant 0 : Z_t \notin D\}.$$

By considering the conformal map $f(z) = (z^2 - 2ia^2)/(z^2 + 2ia^2), z \in D$, or otherwise, determine the distribution of Z_T.

6 State Girsanov's theorem and deduce the Cameron–Martin formula.

Let B be a Brownian motion in $\mathbb{R}$, starting from 0. For which of the following processes is its distribution on $C(\mathbb{R}^+, \mathbb{R})$ given by a density with respect to Wiener measure? Justify your answers.

(a) $B_t + t$,

(b) $B_t - (t \wedge 1)$,

(c) $2B_t$,

(d) $B_t - (t \wedge 1)B_0$.