UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 28 May, 2004 1.30 to 3.30

PAPER 46

STATISTICAL FIELD THEORY

Attempt **TWO** *questions.*

There are **three** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Give an account of the Landau-Ginsberg (LG) theory of phase transitions which should include a discussion of the following points:

(a) The idea of an order parameter;

(b) The distinction between first-order and continuous phase transitions and how their occurrence is predicted in LG theory;

(c) The Maxwell construction, the occurrence of domains and the phenomenon of hysteresis;

(d) The idea of *critical exponents* and how they may be derived;

(e) The features of a tricritical point, how it occurs in LG theory and a brief description of a phase diagram containing a tricritical point.

For a system described by a single scalar field use LG theory to calculate two critical exponents of your choice at an ordinary critical point.

2 The Ising model in D dimensions is defined on a cubic lattice of spacing a with N sites and with spin σ_r on the r-th site. The Hamiltonian is defined in terms of a set of operators $O_i(\{\sigma\})$ by

$$\mathcal{H}(\mathbf{u},\sigma) \;=\; \sum_i u_i O_i(\{\sigma\}) \;,$$

where the u_i are coupling constants with $\mathbf{u} = (u_1, u_2, \ldots)$ and $\sigma_r \in \{1, -1\}$. In particular, $\mathcal{H}$ contains the term $-h\sum_r \sigma_r$ where h is the magnetic field. The partition function is given by

$$\mathcal{Z}(\mathbf{u}, C, N) \;=\; \sum_\sigma \exp(-\beta\mathcal{H}(\mathbf{u},\sigma) - \beta NC) \;.$$

Define the two-point correlation function $G(\mathbf{r})$ for the theory and state how the correlation length ξ parametrizes its behaviour for $r \gg \xi$.

State how the magnetization M and the magnetic susceptibility χ can be determined from the free energy F, and derive the relation which expresses χ in terms of $G(\mathbf{r})$.

Give an short account of how the idea of the Real Space Renormalization Group can be applied to this model including a discussion of the following topics:

(a) The idea of a blocking kernel;

(b) The rôle of fixed points and the scaling hypothesis;

(c) The idea of *relevant* and *irrelevant* operators;

(d) The separation of the expression for the free energy into a singular part $f(\mathbf{u})$ and an inhomogeneous part depending on a function $g(\mathbf{u})$ whose rôle should be defined;

(e) The reason why the inhomogeneous contribution to the free energy may generally be ignored when computing critical exponents.

Close to the critical point of a classical ferromagnet the free energy $F_\pm(t, h, C_0)$ assumes the scaling form

$$F_\pm \;=\; |t|^{2-\alpha}\left(f_\pm\left(\frac{h}{|t|^\Delta}\right) + I_\pm\right) \;,$$

where $t = (T - T_c)/T_c$ and h denotes the magnetic field. What is the significance of the subscript label $\pm$ on these functions?

The following critical exponents are defined:

$$\begin{aligned}
\xi &\sim |t|^{-\nu} & & h = 0 \\
M &\sim |t|^{\beta} & & h = 0,\;\; T < T_C \\
\chi &\sim |t|^{-\gamma} & & h = 0 \\
C_V &\sim |t|^{-\alpha} & & h = 0 \;\; \text{(the specific heat)} \\
M &\sim |h|^{1/\delta} & & T = T_C
\end{aligned}$$

Under suitable assumptions derive the relations

$$\alpha + 2\beta + \gamma \;=\; 2 \;, \quad \alpha \;=\; 2 - D\nu \;, \quad \beta\delta \;=\; \beta + \gamma \;,$$

Question 2 continues on the next page.

where D is the dimension of space.

According to the scaling hypothesis the correlation function takes the form

$$G(\mathbf{r}) \;=\; \frac{1}{|\mathbf{r}|^{D-2+\eta}}\, f_G\left(\frac{|\mathbf{r}|}{\xi(t,h)}\right) \;.$$

What behaviour is expected for the function $f_G(x)$ in the limit $x \to \infty$?

From this formula obtain an expression for the susceptibility and derive the identity

$$\gamma \;=\; (2-\eta)\nu \;.$$

3 Explain what is meant by the *partition function*, Z, of a statistical system and state how Z is related to the *free energy*, F.

A system in D dimensions is described by a Hamiltonian density $\mathcal{H}(\Lambda,\phi)$, where ϕ is a scalar field and Λ is the Ultra-Violet cut-off in units of inverse length. Explain how the dependence of $\mathcal{H}$ on Λ can be such that the large-scale properties of the system are independent of Λ. Why is it advantageous to do this? Also, briefly discuss the rôle of field renormalization in this context. Hence show that, under certain assumptions that should be stated, Landau's approach corresponds to the identification

$$F(M) \;=\; \lim_{\Lambda\to 0} \mathcal{H}(\Lambda, M) \;,$$

where the meaning of the field M is to be defined.

Explain how the Landau approach breaks down at and below a critical dimension, D_c, and calculate D_c for an ordinary critical point and for a tricritical point. You may assume the details of the loop expansion.

[END OF PAPER]