UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 3 June 2002 9 to 11

PAPER 64

STATISTICAL FIELD THEORY

*Attempt **THREE** questions*

*There are **three** questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Give an account of the Landau-Ginsberg(LG) theory of phase transitions which should include a discussion of the following points:

(a) The idea of an order parameter;

(b) The distinction between first-order and continuous phase transitions and how their occurrence is predicted in LG theory;

(c) The idea of *critical exponents* and how they may be derived;

(d) The features of a tricritical point and how it occurs in LG theory;

(e) The notion of critical dimension d_c and why the predictions for critical exponents in LG theory fail for dimension d less than d_c.

For a system described by a single scalar field use LG theory to calculate two critical exponents of your choice at an ordinary critical point.

2 A spin model in D dimensions is defined on a lattice of spacing a with N sites and with spin σ_r on the r-th site. The Hamiltonian is defined in terms of a set of operators $O_i(\{\sigma\})$ by

$$\mathcal{H}(\mathbf{u},\sigma) \;=\; \sum_i u_i O_i(\{\sigma\}) \,,$$

where the u_i are coupling constants with $\mathbf{u} = (u_1, u_2, \ldots)$. The partition function is given by

$$\mathcal{Z}(\mathbf{u}, C, N) \;=\; \sum_\sigma \exp(-\beta\mathcal{H}(\mathbf{u},\sigma) - \beta NC) \,.$$

Explain how a renormalization group (RG) transformation may be defined in terms of a blocking kernel and state how a and N rescale in terms of the RG scale factor b.

The values of the parameter $(\mathbf{u}, C)$ after p blockings are denoted $(\mathbf{u}_p, C_p)$. The RG transformation for $\mathbf{u}$ can be written as

$$\mathbf{u}_p \;\to\; \mathbf{u}_{p+1} \;=\; \mathbf{R}(\mathbf{u}_p) \,.$$

Write down the form of the corresponding RG transformation for $C_p \to C_{p+1}$. What is the physical interpretation of C_p?

Derive the RG equation for the free energy $F(\mathbf{u}_p, C_p)$ and explain how it may be expressed in terms of a singular part $f(\mathbf{u})$ which obeys the RG equation

$$f(\mathbf{u}_0) \;=\; b^{-pD} f(\mathbf{u}_p) \;+\; \sum_j^{p-1} b^{-jD} g(\mathbf{u}_j) \,.$$

What is the origin of the function $g(\mathbf{u})$ which determines the inhomogeneous part of this transformation?

Interpret the behaviour of the RG equations in the neighbourhood of a fixed point in terms of a rescaling of the couplings $\mathbf{u}$ and of the spins $\{\sigma\}$. Explain briefly the concept of *relevant* and *irrelevant* operators.

By considering the behaviour of $f(\mathbf{u}_p)$ in the neighbourhood of a fixed point of the RG equations explain how the critical exponents of a continuous phase transition in the model may be calculated.

The Gaussian model in D dimensions for a real scalar field is defined by the Hamiltonian

$$\mathcal{H} = \frac{1}{2}(\kappa^{-1}(\nabla\phi(\mathbf{x}))^2 + m^2\phi^2(\mathbf{x})) + h\phi(\mathbf{x}) \,,$$

where κ and h are constants.

By defining a suitable thinning transformation show that the critical exponents α and β are given by

$$\alpha = (4-D)/2 \,, \quad \beta = (D-2)/4 \,.$$

3 State what is meant by the correlation length ξ of a scalar field theory and explain briefly why ξ diverges at a second-order phase transition.

A scalar field theory in $4-\epsilon$ dimensions near to a critical point is described by the Hamiltonian density

$$\mathcal{H}(\phi) \;=\; \frac{1}{2}\nabla\phi(\mathbf{x})\cdot\nabla\phi(\mathbf{x}) \;+\; \frac{1}{2}m^2(\Lambda,T)\phi^2(\mathbf{x}) \;+\; \frac{1}{4!}g(\Lambda,T)\phi^4(\mathbf{x}) \;+\; \dots\;,$$

where T is the temperature and Λ is the ultra-violet cutoff. By requiring that the properties of the theory be independent of the choice of Λ show that the dimensionless couplings (u,λ) associated with (m,g) obey renormalization group flow equations, correct to lowest order in ϵ, of the form

$$\begin{aligned}\frac{du^2}{db} &= 2u^2 \;+\; \frac{\Omega_d}{2(2\pi)^d}\,\frac{\lambda}{1+u^2}\\ \frac{d\lambda}{db} &= \epsilon\lambda \;-\frac{3}{2}\frac{\Omega_d}{(2\pi)^d}\,\frac{\lambda^2}{(1+u^2)^2}\;,\end{aligned}$$

where Ω_d is the area of a unit sphere in d dimensions, $b=\log(\Lambda_0/\Lambda)$ and initial conditions are given for $\Lambda=\Lambda_0$.

Treating ϵ as a small positive parameter show that these equations have an infra-red attractive fixed point at

$$\lambda^* \;=\; \frac{16\pi^2\epsilon}{3}\;, \qquad\qquad u^{*2} \;=\; -\frac{\epsilon}{6}\;.$$

Hence show that to lowest order in ϵ

$$\xi \;\sim\; |T-T_c|^{-\nu}\;,$$

where T_c is the critical temperature and $\nu = 1/2+\epsilon/12$.

You may quote the rules of perturbation theory and results from the theory of the renormalization group without derivation.