MATHEMATICAL TRIPOS   Part III

Monday 12 June, 2006   1.30 to 3.30

PAPER 36

SPREAD OF EPIDEMICS AND RUMOURS

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

**You may not start to read the questions
printed on the subsequent pages until
instructed to do so by the Invigilator.**

1 The population consists of only susceptibles and infectives. A susceptible that is infected will stay infectious indefinitely.

We suppose that the population is closed, i.e. $X(t) + Y(t) = n$ for all $t \geqslant 0$ where $X(t)$ and $Y(t)$ denote the number of susceptibles and infectives at time t, with the initial condition $X(0) = an$ and $Y(0) = (1-a)n$, $a \in (0,1)$. At time t, a susceptible individual turns infected at rate $\lambda Y(t)/n$ where $\lambda > 0$.

(a) Let $X_n(t) = X(t)/n$ be the fraction of susceptible individuals at time t. Show that

$$X_n(t) = a - \int_0^t \lambda X_n(s)[1 - X_n(s)]\, ds - \epsilon_n(t),$$

where $\epsilon_n(t)$ is expressed in terms of a standard Poisson process.

(b) Prove that
$$\lim_{n \to \infty} \sup_{0 \leqslant t \leqslant T} |X_n(t) - x(t)| = 0, \quad \text{a.s.}$$

where
$$x(t) = \frac{ae^{-\lambda t}}{1 - a(1 - e^{-\lambda t})}.$$

[Hints: You may assume without proof the following results:

- Let $\mathcal{N}$ be a standard Poisson process and ϵ a positive constant. Then for $T > 0$,

$$\mathbb{P}\left(\sup_{0 \leqslant t \leqslant T} |\mathcal{N}(t) - t| \geqslant T\epsilon \right) \leqslant 2e^{-Th(\epsilon)},$$

where $h(t) = (1+t)\log(1+t) - t$.

- Gronwall's lemma: Let f be a real-valued function satisfying

$$f(t) \leqslant \alpha + \beta \int_0^t f(s)ds, \, t \geqslant 0.$$

Then $f(t) \leqslant \alpha e^{\beta t}$, for $t \geqslant 0$.]

2   Fix $p \in (0,1)$ and let $G(n,p)$ be the Erdös-Rényi random graph with n nodes, where each pair of nodes is connected independently with probability p (here p is constant and does not depend on n).

(a) Show that

$$\mathbb{P}\left(G(n,p) \text{ is disconnected}\right) \leqslant \sum_{s=1}^{[n/2]} \left(n(1-p)^{n/2}\right)^s,$$

and deduce that

$$\lim_{n \to \infty} \mathbb{P}\left(G(n,p) \text{ is connected}\right) = 1.$$

[*Hint: Find an upper bound on the probability that the graph admits an isolated component of size s, for $s \in \{1, \ldots, [n/2]\}$.*]

(b) In the following, we are interested in the diameter of $G(n,p)$ for $p \in (0,1)$.

   (i) For X a random variable in $\{0,1,2,\ldots\}$ show that

$$\mathbb{P}\left(X > 0\right) \leqslant \mathbb{E}(X).$$

   (ii) Let X be the number of pairs of nodes with no common neighbour. Show that

$$\mathbb{E}(X) = \binom{n}{2}(1-p^2)^{n-2}.$$

   (iii) Deduce that

$$\lim_{n \to \infty} \mathbb{P}\left(\text{Diameter of } G(n,p) \leqslant 2\right) = 1.$$

3   We consider an Erdös-Renyi random graph $G(n, p)$ with n nodes, where each pair of nodes is connected independently with probability p. Given i, j, k three distinct nodes in $G(n, p)$, we say that the graph contains the triangle (i, j, k) if the links (i, j), (j, k) and (i, k) are present in $G(n, p)$. Let X be the number of triangles in $G(n, p)$.

(a) Prove that if

$$\lim_{n \to \infty} np = 0\,,$$

then the number of triangles in $G(n, p)$ is zero with probability tending to 1.

[*Hint: Use the following inequality:* $\mathbb{P}(X > 0) \leqslant \mathbb{E}(X)$.]

(b) Assume that

$$\lim_{n \to \infty} np = \infty\,.$$

(i) Note that

$$X = \sum X_{ijk}\,,$$

where $X_{ijk} = 1$ if the triangle ijk is present in $G(n, p)$ and $X_{ijk} = 0$ otherwise, and where the sum is over distinct nodes i, j and k.

Show that

$$\mathbb{E}(X^2) = \sum_{l=0}^{3} \binom{n}{3}\binom{3}{l}\binom{n-3}{3-l}p^{2\binom{3}{2}-\binom{l}{2}}\,.$$

(ii) Using Chebyshev's inequality, establish the following inequality

$$\mathbb{P}(X = 0) \leqslant \frac{\mathrm{Var}(X)}{(\mathbb{E}X)^2}\,,$$

where $\mathrm{Var}(X)$ is the variance of X.

(iii) Prove that

$$\lim_{n \to \infty} \mathbb{P}(\text{there is at least one triangle in } G(n, p)) = 1\,.$$

4   We consider an undirected graph $G = (V, E)$ where $V = \{1, \ldots, n\}$ is the set of nodes and $E \subset V \times V$ is the set of edges and let A be its adjacency matrix, i.e. $A_{ij} = A_{ji} = 1$ if i and j are connected in G (i.e. $(i, j) \in E$) and 0 otherwise. We assume that each node can be in one of three states: susceptible, infected or removed, and that initially there is at least one infected node.

The evolution of the epidemic is described by the following discrete-time model: an infected node stays infected for a unit of time during which it infects each of its neighbours with probability β and is then removed.

(a) Let $X_i(k)$ be the indicator that node i is infected at time k, and $Y_i(k)$ the indicator that node i is removed at time k. Show that

$$\mathbb{P}\left(Y_i(\infty) = 1\right) = \mathbb{P}\left(\text{node } i \text{ ever gets infected}\right) \leqslant \sum_{k=0}^{\infty} \sum_{j=1}^{n} \beta^k (A^k)_{ij} X_j(0).$$

(b) Deduce that

$$\mathbb{E}\left[\sum_{i=1}^{n} Y_i(\infty)\right] \leqslant \sum_{k \geqslant 0} (1, \ldots, 1)(\beta A)^k X(0),$$

where

$$X(0) = \begin{pmatrix} X_1(0) \\ \vdots \\ X_n(0) \end{pmatrix}.$$

(c) Let ρ be the spectral radius of the adjacency matrix A (the eigenvalue of A with the largest absolute value) and suppose that $\beta \rho < 1$. Show that

$$\mathbb{E}\left[\sum_{i=1}^{n} Y_i(\infty)\right] \leqslant \frac{1}{1 - \beta \rho} \sqrt{n \sum_{i=1}^{n} X_i(0)}.$$

Deduce a sufficient condition for a small outbreak.

END OF PAPER

Paper 36