MATHEMATICAL TRIPOS Part III

Tuesday 10 June 2008 9.00 to 11.00

PAPER 55

SOLITONS AND INSTANTONS

Attempt no more than **THREE** *questions.*

There are **FOUR** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider the following action for sine-Gordon equation on a space-time $\mathbb{R}^2$, in the presence of an external electromagnetic potential $A_0 dt + A_1 dx$, (where $A_0(t,x)$ and $A_1(t,x)$ are given and smooth):

$$S[\theta, \theta_t] = \int \Big[\frac{1}{2}\theta_t^2 - \frac{1}{2}\theta_x^2 - (1 - \cos\theta) - A_0\theta_x + A_1\theta_t\Big] dtdx.$$

Show that the corresponding equation of motion (the Euler-Lagrange equation) is

$$\theta_{tt} - \theta_{xx} + \sin\theta + E = 0$$

where E is the electric field $E \equiv \partial_t A_1 - \partial_x A_0$. In the case that E is everywhere zero, this equation has static "kink" solutions given by $\theta_K(x) \equiv 4\arctan e^x$.(You need not prove this). Deduce that the formulas

$$\theta(t,x) = \theta_K\big(\gamma(x - X(t))\big), \qquad \theta_t(t,x) = -\gamma u \theta_K'\big(\gamma(x - X(t))\big)$$

define a solution as long as $X(t) = X_0 + ut$, for any constants $X_0 \in \mathbb{R}, u \in (-1, +1)$, and with $\gamma = (1 - \dot{X}^2)^{-\frac{1}{2}}$ and E still everywhere zero.

Now, assuming that $A_1 \equiv 0$ and $A_0(t,x) = xf(t)$, calculate the following action functional:

$$S_{eff}[X, \dot{X}] \equiv S\Big[\theta_K\big(\gamma(x - X)\big), -\gamma u \theta_K'\big(\gamma(x - X)\big)\Big]$$

by substituting the above exact solution into S, and show that it is equal to:

$$S_{eff}[X, \dot{X}] = -\int \Big[8\sqrt{1 - \dot{X}^2} + 2\pi X f(t)\Big] dt.$$

(This "effective" action could be expected to give an approximation to the motion of kinks under the influence of the external potential for small f.) Obtain the equation of motion for X which follows from S_{eff}, and comment on your answer.

2 Consider the static abelian Higgs model on the spatial domain $\Sigma = \{z = x + iy \in \mathbb{C} : |z|^2 = x^2 + y^2 < 1\}$, with metric

$$g = e^{2\rho}(dx^2 + dy^2) = \frac{8(dx^2 + dy^2)}{(1 - x^2 - y^2)^2}.$$

Φ is a section of a complex line bundle L over Σ, and $D = (D_x, D_y) = (\partial_x - iA_1, \partial_y - iA_2)$ is the covariant derivative acting on sections of L, determined by the connection one-form $A = A_1 dx + A_2 dy$. The associated magnetic field B is defined by $B = e^{-2\rho}(\partial_x A_2 - \partial_y A_1)$.

The energy is

$$V(A, \Phi) = \frac{1}{2}\int_\Sigma \Big[e^{-2\rho}(\partial_x A_2 - \partial_y A_1)^2 + |D_x\Phi|^2 + |D_y\Phi|^2 + \frac{e^{2\rho}}{4}(1 - |\Phi|^2)^2 \Big] dxdy.$$

Derive the Bogomolny decomposition for $V(A, \Phi)$ and hence obtain a lower bound for V which depends on N, the degree (or winding number), of Φ. Hence, or otherwise, show that for $N = 1$ this lower bound is achieved by the following Φ:

$$\Phi = \frac{2z}{1 + |z|^2}$$

for a particular connection A, which should be given explicitly.

3 Explain the Derrick argument, and its consequences for the non-existence of static solitons, in theories describing fields $\phi : \mathbb{R}^n \to \mathbb{R}^l$ with energy

$$V(\phi) = \int_{\mathbb{R}^n} \Big[\frac{1}{2}|\nabla\phi|^2 + U(\phi) \Big] d^n x,$$

where $U \geqslant 0$.

Explain why this argument does not rule out the existence of solitons for (i) the Yang-Mills-Higgs energy on $\mathbb{R}^3$, (ii) the $\sigma-$ model (or harmonic map) energy on $\mathbb{R}^2$.

Consider a field theory for $\phi : \mathbb{R}^{1+n} \to \mathbb{C}$ determined by an action of the form

$$S(\phi) = \int_{\mathbb{R}^{1+n}} \Big[\frac{1}{2}|\phi_t|^2 - \frac{1}{2}|\nabla\phi|^2 - U(\phi) \Big] dt d^n x.$$

You may assume that $U = U(|\phi|^2)$.

Derive the corresponding equation of motion (the Euler-Lagrange equation). Define a "non-topological soliton " for this equation and explain why the Derrick argument does not rule out the existence of such solutions for $n > 1$. Give, without proof, an example of a potential U for which such solutions do exist.

4 Let $A = A_1 dx^1 + A_2 dx^2 + A_3 dx^3$ be an $su(2)-$valued one-form on $\mathbb{R}^3$ defining a covariant derivative operator $D = \nabla + [A, \cdot]$ on $su(2)-$valued functions $\Phi : \mathbb{R}^3 \to su(2)$. Define the corresponding curvature two-form $F = \sum_{j<k} F_{jk} dx^j \wedge dx^k$. Use the inner product $\langle \Phi, \Psi \rangle = -\frac{1}{2}\mathrm{tr}\Phi\Psi$ and corresponding norm $|\Phi|^2 = \langle \Phi, \Phi \rangle$. Show that

$$\frac{\partial}{\partial x^j}\langle \Phi, \Psi \rangle = \langle D_j \Phi, \Psi \rangle + \langle \Phi, D_j \Psi \rangle.$$

Derive the equations of motion (the Euler-Lagrange equations) for the Yang-Mills-Higgs energy functional on $\mathbb{R}^3$:

$$V(A, \Phi) = \int_{\mathbb{R}^3} v(A, \Phi)\, d^3x \qquad v(A, \Phi) = |B|^2 + |D\Phi|^2,$$

where $B_i = \frac{1}{2}\epsilon_{ijk} F_{jk}$. Show that if A, Φ solve the equations $B_i = D_i \Phi$, $i = 1, 2, 3$ then A, Φ also solve the equations of motion just derived. Assuming still that $B_i = D_i \Phi$, $i = 1, 2, 3$, find a formula for $\Delta |\Phi|^2$, where $\Delta = \sum_{j=1}^{3} \frac{\partial^2}{\partial x^{j\,2}}$, in terms of the energy density $v(A, \Phi)$.

END OF PAPER