UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 9 June, 2005 1.30 to 3.30

PAPER 56

SOLITONS AND INSTANTONS

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $\phi : \mathbb{R}^D \longrightarrow \mathbb{R}$ be a static scalar field in D spatial dimensions with energy functional

$$E(\phi) = \int_{\mathbb{R}^D} \mathrm{d}^D x \left(\tfrac{1}{2}|\nabla\phi|^2 + U(\phi) \right),$$

where $U(\phi) \geqslant 0$. Use the Derrick scaling argument to demonstrate that a configuration ϕ can have finite energy and be a critical point for $E(\phi)$ only if $D = 1$ and

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = \pm\frac{\mathrm{d}W}{\mathrm{d}\phi},$$

for some W which should be determined. What are the topological conserved quantities in this case?

Explain how the restriction $D = 1$ can be overcome by introducing gauge fields.

2 A map $\phi : \mathbb{R}^2 \longrightarrow S^2$ is defined by a field $\phi^a(x^i)$ subject to the constraint $\phi^a\phi^a = 1$ ($i = 1,2$; $a = 1,2,3$; and the summation convention applies regardless of the position of indices). Show that the energy density

$$\mathcal{E} = \tfrac{1}{2}\partial_i\phi^a\partial_i\phi^a$$

results in field equations

$$\triangle\phi^a - (\phi^b\triangle\phi^b)\phi^a = 0,$$

where $\triangle = \partial_i\partial_i$ is the Laplacian on $\mathbb{R}^2$.

Explain why maps $\phi : \mathbb{R}^2 \longrightarrow S^2$ for which the total energy $\int \mathcal{E}\mathrm{d}^2x$ is finite can be extended to a one-point compactification of $\mathbb{R}^2$, i.e. to maps $\phi : S^2 \longrightarrow S^2$. Show that, for such maps, the total energy is bounded from below by $4\pi|Q|$, where

$$Q = \frac{1}{8\pi}\int \varepsilon^{ij}\varepsilon^{abc}\phi^a\partial_i\phi^b\partial_j\phi^c \,\mathrm{d}^2x$$

(ε is the totally antisymmetric tensor in two or three dimensions with $\varepsilon^{12} = \varepsilon^{123} = 1$). What does this have to do with the topological degree of a map?

[*Hint* : $\int(\partial_i\phi^a \pm \varepsilon_{ij}\varepsilon^{abc}\phi^b\partial_j\phi^c)(\partial_i\phi^a \pm \varepsilon_{ij}\varepsilon^{abc}\phi^b\partial_j\phi^c)\mathrm{d}^2x \geqslant 0.$]

3 Formulate the Euclidean anti–self–dual Yang–Mills (ASDYM) equations as compatibility conditions for an overdetermined system of linear equations.

Now impose three translational symmetries, and choose a suitable gauge to reduce the ASDYM equations to the system of ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}A_a = \frac{1}{2}\varepsilon_{abc}[A_b, A_c]$$

where a, b, c run over 1 to 3, and $A_a = A_a(t)$ take values in the Lie algebra of some Lie group. By considering the commutator $[A(\lambda), B(\lambda)]$, where

$$A(\lambda) = (A_1 + iA_2) + 2A_3\lambda - (A_1 - iA_2)\lambda^2\,,$$

$$B(\lambda) = -iA_3 + i(A_1 - iA_2)\lambda\,, \qquad \lambda \in \mathbb{C}P^1\,,$$

show that the coefficients of the polynomials $\mathrm{Tr}(A(\lambda)^p)$ are independent of t for any p.

4 Define the projective twistor space corresponding to the complexified Minkowski space $M_{\mathbb{C}}$.

Write an essay on the inverse Ward correspondence (from holomorphic vector bundles over the projective twistor space to solutions of the anti–self–dual Yang–Mills equations on $M_{\mathbb{C}}$).

END OF PAPER