UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 8 June, 2004 9 to 11

PAPER 51

SOLITONS AND INSTANTONS

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $\phi = \phi(x,t)$ be a scalar field and let $U = U(\phi) \geqslant 0$. Define the energy of solutions to the Euler–Lagrange equations with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\phi_t^2 - \frac{1}{2}\phi_x^2 - U(\phi)$$

and find the Bogomolny equations satisfied by the minimal energy configurations.

Assume that $U = (1/2)\phi^2(\phi^2 - \beta^2)^2$, where $\beta \in \mathbb{R}$. How many static kink solutions are there? Find a moving kink solution for the model if $\beta \neq 0$.

2 Let $(A, \Phi) : \mathbb{R}^{3,1} \longrightarrow \mathbf{su}(2)$ be the Yang–Mills potential and the Higgs field in the adjoint representation respectively, and let $F_{\mu\nu} = [D_\mu, D_\nu]$ be the curvature of the connection $D_\mu = \partial_\mu + A_\mu$.

Find the Euler–Lagrange equations associated to the Lagrangian density

$$\mathcal{L} = \frac{1}{4}Tr(F_{\mu\nu}F^{\mu\nu}) - \frac{1}{2}Tr((D_\mu\Phi)(D^\mu\Phi)) + \frac{m^2}{2}Tr(\Phi^2)$$

and verify that any solution to the Bogomolny equations

$$B_i = D_i\Phi$$

satisfies the static field equations with $m = 0$. Here $B_k := (1/2)\varepsilon_{ijk}F_{ij}$, and F_{ij} are components of the spatial curvature.

3 Let $\omega_i, i = 1, 2, 3$ be a basis of self–dual two–forms in $\mathbb{R}^4$. Show that a Lie algebra valued one–form A such that

$$\omega_i \wedge F = 0, \qquad \text{where} \qquad F = \mathrm{d}A + A \wedge A \tag{1}$$

satisfies the Euclidean Yang–Mills equations.

By considering the metric in null coordinates or otherwise show that (1) arises as the compatibility conditions for an overdetermined system of linear equations.

Now assume that A takes its values in $\mathbf{u(1)}$, and deduce that in null coordinates $w, z \in \mathbb{C}$ such that the metric is $\mathrm{d}s^2 = 2(\mathrm{d}z\mathrm{d}\overline{z} + \mathrm{d}w\mathrm{d}\overline{w})$ equations (1) become

$$\partial_w A_z - \partial_z A_w = 0, \qquad \partial_{\overline{w}} A_{\overline{z}} - \partial_{\overline{z}} A_{\overline{w}} = 0,$$

$$\partial_z A_{\overline{z}} - \partial_{\overline{z}} A_z + \partial_w A_{\overline{w}} - \partial_{\overline{w}} A_w = 0.$$

Deduce the existence of scalar functions u and v such that

$$A = \partial_w u \; \mathrm{d}w + \partial_z u \; \mathrm{d}z + \partial_{\overline{w}} v \; \mathrm{d}\overline{w} + \partial_{\overline{z}} v \; \mathrm{d}\overline{z}\,,$$

and by making a gauge transformation reduce (1) to the Laplace equation

$$\frac{\partial^2 f}{\partial z \partial \overline{z}} + \frac{\partial^2 f}{\partial w \partial \overline{w}} = 0,$$

where $f = f(w, z, \overline{w}, \overline{z})$.

4 Show that a trivial rank(k) holomorphic vector bundle $E \to Z$ admits holomorphic splitting matrices $H_\alpha : U_\alpha \to \mathrm{GL}(k, \mathbb{C})$ such that

$$F_{\alpha\beta} = H_\beta H_\alpha^{-1},$$

where $F_{\alpha\beta}$ is a holomorphic patching matrix defined with respect to some coordinate charts U_α of a complex manifold Z.

Assume that $E \to \mathcal{PT}$ is a rank one holomorphic vector bundle over the twistor space of the complexified Minkowski space $M_{\mathbb{C}}$ which is trivial on each twistor line, and show how E gives rise to solutions of anti–self–dual (ASD) Maxwell equations on $M_{\mathbb{C}}$. Consider a covering of $\mathcal{PT}$ by two open sets, and show that the ASD Maxwell potential corresponding to the patching function $F_{01} = e^f$ is in some gauge given by

$$A_{BB'} = \frac{1}{2\pi i} \oint_\Gamma \frac{\iota_{B'}}{(\iota_{C'} \pi^{C'})} \frac{\partial f}{\partial \omega^B} \pi_{D'} \mathrm{d}\pi^{D'},$$

where $f \in H^1(\mathcal{PT}, \mathcal{O})$, ι is a constant spinor, and $\Gamma \subset \mathbb{CP}^1$ is a real contour in a twistor line $\omega^A = x^{AA'} \pi_{A'}$. Verify that the corresponding field $\mathrm{d}A$ is indeed ASD.