MATHEMATICAL TRIPOS Part III

Monday 9 June 2008 1.30 to 3.30

PAPER 86

SLOW VISCOUS FLOW

Attempt no more than **THREE** *questions.*

There are **FIVE** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Find the Stokes flow outside a small sphere of radius a translating at velocity $\mathbf{V}$ through an infinite viscous fluid.

[Hint: the Papkovich-Neuber representation is

$$\mathbf{u} = \nabla\phi + \nabla(\mathbf{x}\cdot\mathbf{A}) - 2\mathbf{A} \quad \textit{with} \quad p = 2\mu\nabla\cdot\mathbf{A}.]$$

Now consider two well separated slightly unequal spheres of radii a_1 and a_2 and isolated sedimentation speeds V_1 and V_2, with $|a_1V_1 - a_2V_2| \ll (a_1+a_2)|V_1-V_2| \ll (a_1+a_2)(V_1+V_2)$. Using only the leading order $O(a/r)$ hydrodynamic interaction between the spheres, derive an equation for the separation $\mathbf{r} = \mathbf{x_1} - \mathbf{x_2}$ of the centres of the spheres. Reduce this to the polar form (with θ the angle to the upward vertical)

$$\begin{aligned} \dot{r} &= -\left((V_1 - V_2) + \frac{3(a_2V_2 - a_1V_1)}{2r}\right)\cos\theta + O(a^3/r^3), \\ r\dot{\theta} &= \left((V_1 - V_2) + \frac{3(a_2V_2 - a_1V_1)}{4r}\right)\sin\theta + O(a^3/r^3). \end{aligned}$$

[You may assume the error terms are in fact $O((a_1^2+a_2^2)(a_2V_2-a_1V_1)/r^3)$ and so negligible even when $(a_2V_2 - a_1V_1)$ is small.]

Show that if $(V_1 - V_2)$ and $(a_2V_2 - a_1V_1)$ are of the same sign, then one sphere overtakes the other and the separation eventually increases indefinitely. On the other hand if they have opposite signs, then one sphere can be attracted to a stable equilibrium vertically above the other. Comment on reversibility in this latter case.

2 Write down the Green's function (Stokelet) for Stokes flow.

Sketch the derivation of the *slender-body approximation* for the force $\mathbf{f}(s)$ exerted on the fluid per unit length

$$\mathbf{f}(s) \sim \frac{2\pi\mu}{\ln(L/R)}\left(2\mathbf{I} - \mathbf{X}'\mathbf{X}'\right)\cdot\dot{\mathbf{X}}.$$

Now consider a thin rigid wire in the form of a helix

$$\mathbf{X}(\theta) = (a\cos\theta, \sin\theta, b\theta) \quad \text{for} \quad -\pi \le \theta \le \pi,$$

with arc-length $s = \theta\sqrt{a^2+b^2}$. Find the z-component of the total force when the wire rotates at angular velocity Ω about the z-axis. Find the z-component of the total couple when the wire translates at velocity W in the z-direction. Why are the coefficients identical?

3 Consider the flow $\mathbf{U}$ past a fixed sphere of radius a at small non-zero Reynolds number. Show that at distances $r = O(\nu/U)$ the inertial terms cannot be neglected.

Write down the Oseen equations for small disturbances $\mathbf{u}$ to the uniform flow $\mathbf{U}$ far from the sphere. Using the representation

$$\mathbf{u} = \nabla\phi + \nu\nabla\chi - \mathbf{U}\chi \quad \text{and taking} \quad \chi = he^{\mathbf{U}\cdot\mathbf{x}/2\nu},$$

find an Oseen flow which decays at large distances and for $r \ll \nu/U$

$$\mathbf{u} \sim -\left(\mathbf{U}\frac{3a}{4r} + \mathbf{x}(\mathbf{U}\cdot\mathbf{x})\frac{3a}{4r^3}\right).$$

Show that ϕ represents a point mass source of strength $6\pi\mu aU$, and χ represents a wake.

4 A sphere of radius R falls under gravity down the axis of a vertical cylinder of radius $R+d$, with $d \ll R$, filled with a viscous liquid. Show that the fall speed is

$$\frac{8\sqrt{2}\Delta\rho g d^2}{27\mu}\left(\frac{d}{R}\right)^{1/2}.$$

[Hint: In a two-dimensional problem of a fixed cylinder of radius a at a small distance h from a fixed flat wall, find the pressure drop Δp required to drive a volume flux Q (per unit length in the 3rd dimension).]

How does the fall speed of the sphere in the cylinder compare with the sedimentation speed of a sphere of radius d in an unbounded liquid?

If the sphere were not centrally positioned in the cylinder, would it fall faster or slower?

[You may assume that if $0 < \epsilon < 1$ and $\alpha > 0$ then $\int_0^{2\pi}(1+\epsilon\cos\theta)^\alpha\, d\theta$ increases with ϵ.]

5 Consider a fixed volume V of viscous liquid spreading under gravity confined to a horizontal shallow V-shaped channel, *i.e.* with boundary $z = \alpha|y|$ for $-\infty < x < \infty$, and with $0 < \alpha \ll 1$.

Find a nonlinear diffusion equation for the depth of liquid $h(x,t)$ which governs the spreading.

Find the scalings for the variation in time of the typical depth H and the length L along the channel (in the x-direction).

Find a similarity solution for $h(x,t)$. *[In your answer, let $I(\beta) = \int_0^\pi \cos^\beta\theta\, d\theta$.]*

END OF PAPER