UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 31 May 2007 1.30 to 4.30

PAPER 78

SLOW VISCOUS FLOW

*Attempt at most **THREE** questions.*

Substantially complete answers will be regarded more favourably than fragmentary answers.

It is possible to gain a quality mark of alpha minus for the paper with substantially complete answers to two questions.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

The Papkovich–Neuber representation of Stokes flow is

$$\mathbf{u} = \boldsymbol{\nabla}(\mathbf{x} \cdot \boldsymbol{\Phi} + \chi) - 2\boldsymbol{\Phi}, \quad p = 2\mu \boldsymbol{\nabla} \cdot \boldsymbol{\Phi}, \quad \text{where } \nabla^2 \chi = 0 \text{ and } \nabla^2 \boldsymbol{\Phi} = \mathbf{0}.$$

1 State and prove the Minimum Dissipation Theorem for Stokes flow, making it clear which flows are compared by the theorem.

The annular region between two concentric rigid spheres of radii a and b (with $b > a$) is filled with fluid of viscosity μ. The outer sphere is held stationary, while the inner sphere is made to rotate with angular velocity $\boldsymbol{\Omega}$. Determine the fluid velocity.

Show explicitly that the Papkovich–Neuber solution gives $p = 0$. Explain how this result could also be obtained without detailed calculation from simple properties of Stokes flow.

Calculate the stress field in the fluid. Deduce that the couple $\mathbf{G}$ that must be applied to the inner sphere to maintain the motion is given by

$$\mathbf{G} = \frac{8\pi\mu a^3 b^3 \boldsymbol{\Omega}}{b^3 - a^3} \ .$$

Comment on the form of this result for $a \ll b$ and for $b - a \ll a$.

A number of force-free, couple-free rigid particles are added to the fluid between the spheres, but the concentric position and relative angular velocity $\boldsymbol{\Omega}$ of the inner sphere are maintained by application of the necessary force and couple to the inner sphere. Show that the component of the couple in the direction of $\boldsymbol{\Omega}$ is increased, being careful to explain each step of the argument.

2 Fluid of viscosity μ surrounds a very long axisymmetric fluid bubble of radius $r = a(z,t)$ (in cylindrical polar coordinates) and viscosity $\lambda\mu$, where $\lambda \ll 1$. The bubble radius a varies slowly with z, so that $|\partial a/\partial z| \ll 1$. The ends of the bubble are far enough away to be neglected, and the flow is driven only by surface tension.

Explaining any approximations made to the boundary conditions on $r = a$, show that the external flow is approximately radial and that the pressure $P(z,t)$ in the bubble is given by

$$P = \frac{2\mu}{a}\frac{\partial a}{\partial t} + \frac{\gamma}{a}, \tag{1}$$

where γ is the coefficient of surface tension.

(a) Assume that λ is sufficiently small that the pressure gradient inside the bubble can be neglected. Show that if $a(z,t) = f(t) + \sqrt{2}g(t)\sin(kz)$, where k is a constant, $ak \ll 1$ and $f > \sqrt{2}g > 0$, then

$$P(t) = \frac{\gamma f(t)}{\alpha_0}, \qquad \frac{df}{dt} = -\frac{\gamma}{2\mu}\frac{g^2(t)}{\alpha_0} \quad \text{and} \quad \frac{dg}{dt} = \frac{\gamma}{2\mu}\frac{f(t)g(t)}{\alpha_0},$$

where α_0 is a constant to be determined in terms of $f(0)$ and $g(0)$.

Find the linear growth rate of a small sinusoidal disturbance to a uniform cylinder of radius a_0.

Show also that the minimum radius $a_{\min}$ continues to decrease monotonically as the disturbance grows in the nonlinear regime. [*You do not need to solve the equations for f and g explicitly.*]

(b) Show that the pressure gradient inside the bubble cannot be neglected when $a_{\min}k \ll \lambda$. Using lubrication theory, show that the evolution is then given by

$$2a\frac{\partial a}{\partial t} = \frac{1}{8\lambda\mu}\frac{\partial}{\partial z}\left\{a^4\frac{\partial}{\partial z}\left(\frac{2\mu}{a}\frac{\partial a}{\partial t} + \frac{\gamma}{a}\right)\right\}\ . \tag{2}$$

Just before the bubble breaks at some time t^* and position z^*, there is a local similarity solution of (2) in which $a \propto (t^* - t)^p$ and $z - z^* \propto (t^* - t)^q$ as $t \to t^*$. Use scaling estimates of the different terms to determine the exponents p and q, and to define dimensionless similarity variables $A(\zeta)$ and ζ such that

$$\zeta AA' - A^2 = \frac{1}{16}\left[A^4\left(\frac{2\zeta A' + 1}{A}\right)'\right]'.$$

[*You are not required to solve this equation.*]

3 Insoluble surfactant with concentration $C(x,t)$ resides on the surface of a thin layer of fluid of thickness $h(x,t)$, viscosity μ and density ρ that lies on a rigid horizontal surface. Diffusion of surfactant is negligible, and the coefficient of surface tension is given by $\gamma(C) = \gamma_0 - AC$, where γ_0 and A are constants. The variations of h and C are such that lubrication theory is applicable throughout.

Explain why

$$\frac{\partial C}{\partial t} + \frac{\partial [u(h)C]}{\partial x} = 0,$$

where $u(h)$ is the horizontal velocity at the surface. Show that

$$\frac{\partial h}{\partial t} = \frac{A}{2\mu}\frac{\partial}{\partial x}\left(h^2\frac{\partial C}{\partial x}\right) + \frac{\rho g}{3\mu}\frac{\partial}{\partial x}\left(h^3\frac{\partial h}{\partial x}\right) - \frac{1}{3\mu}\frac{\partial}{\partial x}\left(h^3\frac{\partial}{\partial x}\left(\gamma\frac{\partial^2 h}{\partial x^2}\right)\right), \tag{3}$$

and obtain the corresponding equation for the evolution of C.

Assume that the hydrostatic and capillary pressure gradients are both negligible. A fixed mass $M = \int C\,dx$ of surfactant is released at $x = 0$ and $t = 0$ onto a layer that initially has uniform thickness h_0 and $C = 0$. Use scaling arguments to show that the extent $-x_N \leqslant x \leqslant x_N$ of the spreading pool of surfactant satisfies $x_N(t) \propto t^{1/3}$.

Deduce the form of the similarity solution and derive two ordinary differential equations and two integral constraints that are satisfied by the dimensionless similarity functions $H(\eta)$ and $\Gamma(\eta)$ over the range $0 \leqslant \eta \leqslant \eta_N$ (assuming symmetry about $\eta = 0$). Solve the differential equations to show that H and Γ are linear functions of η, and use the integral constraints to show that

$$x_N = (6MAh_0t/\mu)^{1/3}.$$

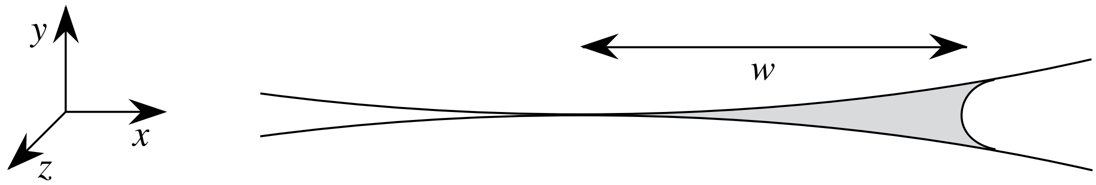
Explain why the hydrostatic and capillary terms in (3) cannot both be negligible near $x = x_N$. Let Δ be the width of the region where at least one of these terms is significant. Use scaling arguments to show that when $g = 0$

$$\Delta \sim (\gamma_0 h_0^2 x_N^2/AM)^{1/3} \propto t^{2/9}\ .$$

[*Assume that* $\partial C/\partial x$ *continues to scale like* M/x_N^2.]

Find the corresponding result for Δ when $g \neq 0$ and $\Delta^2 \gg \gamma_0/g$. For this case find the time t^* when $\Delta \sim x_N$. Explain briefly what happens to the fluid layer when $t \gg t^*$?

4 Two infinite rigid cylinders of radius a are parallel and touching along the z-axis. A small amount of viscous fluid occupies the cusp-shaped region on one side of the cylinders, and the wetting properties of the fluid are such that the meniscus is tangent to the cylinders at the contact points, as shown in the diagram. The width of fluid $w(z,t)$ satisfies $w \ll a$ and varies slowly in the axial direction z.



Making appropriate geometrical approximations, show that the cross-sectional area of fluid is proportional to w^3 and that the curvature of the semicircular meniscus is proportional to w^{-2}. Find the constants of proportionality.

Gravity is negligible and the fluid is drawn (in both directions) along the cusp between the cylinders by the variation in the capillary pressure. By integrating the flux over the cross-section, derive the equation

$$\frac{\partial w^3}{\partial t} = \frac{\gamma}{7\mu a}\frac{\partial}{\partial z}\left(w^4 \frac{\partial w}{\partial z}\right)$$

for the evolution of $w(z,t)$.

Obtain a similarity solution for the spread of a small fixed volume V of fluid placed in the cusp at $z=0$ and $t=0$. In particular, show that the location of one tip of the flow is given by

$$z_N(t) = \left(\frac{8Va}{\pi}\right)^{1/4}\left(\frac{8\gamma t}{21\mu a}\right)^{3/8}.$$

[*Note that* $\displaystyle\int_{-\pi/2}^{\pi/2} \sin^4\theta \, d\theta = \frac{3\pi}{8}$.]

Suppose that gravity is no longer negligible and that the cylinders are now placed, still touching, with their axes vertical in a large bath of fluid whose free surface is at $z=0$. Find $w(z)$ for large z after the flow has come to rest.

END OF PAPER