UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 9 June, 2005 9 to 11

PAPER 82

SEISMIC WAVES

Attempt up to ***FOUR*** *questions.*

Full marks will be awarded for the equivalent of complete answers to **two** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) From the equations for 1-dimensional elastic waves

$$\rho \frac{\partial v}{\partial t} = \frac{\partial \sigma}{\partial x}, \qquad \frac{\partial \sigma}{\partial t} = \mu \frac{\partial v}{\partial x}, \qquad \text{with} \qquad \beta = \sqrt{\frac{\mu}{\rho}},$$

show that in uniform media there are two independent solutions (v_+, σ_+) and (v_-, σ_-) satisfying

$$\sigma_+ = -\rho\beta v_+, \qquad \sigma_- = \rho\beta v_-,$$

with energy fluxes $S = -v\sigma$ in the positive and negative x-directions respectively.

(b) Convert the above differential equations for continuous media into the following equations for ϕ_+ and ϕ_- such that

$$v = \frac{1}{\sqrt{\rho\beta}} (\phi_+ + \phi_-), \qquad \sigma = -\sqrt{\rho\beta} (\phi_+ - \phi_-) :$$

$$\frac{\partial \phi_+}{\partial x} + \frac{1}{\beta} \frac{\partial \phi_+}{\partial t} = \frac{1}{2} \left\{ \frac{\partial}{\partial x} \ln (\rho\beta) \right\} \phi_-,$$
$$\frac{\partial \phi_-}{\partial x} - \frac{1}{\beta} \frac{\partial \phi_-}{\partial t} = \frac{1}{2} \left\{ \frac{\partial}{\partial x} \ln (\rho\beta) \right\} \phi_+.$$

(c) For time-harmonic waves with angular frequency ω and for a continuously differentiable monotonic variation in seismic impedance $\rho\beta$ between x_a and x_b $(> x_a)$ such that

$$|\omega| \int_{x_a}^{x_b} \frac{dx}{\beta} \ll \text{both } 1 \text{ and } \frac{1}{2} \left| \ln \left(\frac{\rho_b \beta_b}{\rho_a \beta_a} \right) \right|,$$

where $\rho_a\beta_a$ and $\rho_b\beta_b$ are the impedances at x_a and x_b, use the differential equations for ϕ_+ and ϕ_- to find reflection and transmission coefficients R_{aa}, T_{ab}, T_{ba}, R_{bb} depending on only $\rho_a\beta_a$ and $\rho_b\beta_b$ such that to zeroth-order in frequency

$$\begin{pmatrix} \phi_- (x_a) \\ \phi_+ (x_b) \end{pmatrix} = \begin{pmatrix} R_{aa} & T_{ab} \\ T_{ba} & R_{bb} \end{pmatrix} \begin{pmatrix} \phi_+ (x_a) \\ \phi_- (x_b) \end{pmatrix}.$$

(d) Relate your answer to part (c) to corresponding expressions for reflection and transmission at a discontinuity in the medium, and show that in this zeroth-order in frequency limit the total time-averaged energy flux $\frac{1}{2} \left[|\phi_- (x_a)|^2 + |\phi_+ (x_b)|^2 \right]$ out of the region is the same as the total time-averaged energy flux $\frac{1}{2} \left[|\phi_+ (x_a)|^2 + |\phi_- (x_b)|^2 \right]$ into the region. Use the differential equations for ϕ_+ and ϕ_- to demonstrate that for time-harmonic waves

$$\frac{d}{dx} \left[|\phi_+ (x)|^2 - |\phi_- (x)|^2 \right] = 0$$

and, hence, that this equality of energy fluxes into and out of the region is satisfied exactly at all frequencies.

2 **Either, (a)** Describe, defining all required terms, all the steps in deriving the transport equation for the leading-order P-wave wavefield discontinuity in the form

$$0 = \frac{\partial}{\partial x_q}\left(\rho\alpha\left(a^{[k]}\right)^2 n_q\right)$$

for a continuous istropic medium, where $a^{[k]}$ is the amplitude of the discontinuity at the wavefront and $k \geqslant 2$ denotes the order of the time derivative of displacement involved.

Or (b) For the case of a uniform isotropic superficial layer of thickness h over a uniform isotropic halfspace, derive the dispersion relationship for Love waves

$$\tan\left(|\omega|\, h\sqrt{\frac{1}{\beta_0^2} - \frac{1}{c_L^2}}\right) = \frac{\mu\sqrt{\frac{1}{c_L^2} - \frac{1}{\beta^2}}}{\mu_0\sqrt{\frac{1}{\beta_0^2} - \frac{1}{c_L^2}}}$$

using both of the following two approaches:

(i) direct solution of the equations for SH waves, and

(ii) derivation of the condition for constructive interference of SH waves multiply reflected in the superficial layer, including derivation of the reflection coefficients involved.

Here ω is the angular frequency, c_L is the Love wave speed, β_0 and β ($> \beta_0$) are the S wave speeds in the layer and the halfspace, and μ_0 and μ are the corresponding shear moduli.

(iii) Explain the significance of the frequencies where $c_L = \beta$.

[**Do not attempt both parts of this question.**]

3 In a bounded perfectly elastic domain $\mathcal{D}$, with boundary $\partial\mathcal{D}$, density $\rho(\mathbf{x})$ and stiffness tensor $c_{ijpq}(\mathbf{x})$, consider the Fourier components $\widehat{u}_i(\mathbf{x},\omega)\,e^{-i\omega t}$ for real-valued frequencies ω of a real-valued displacement field $u_i(\mathbf{x},t)$ associated with real-valued body forces $f_i(\mathbf{x},t)$, with corresponding Fourier components $\widehat{f}_i(\mathbf{x},\omega)\,e^{-i\omega t}$. Each $\widehat{u}_i(\mathbf{x},\omega)$ satisfies

$$0=\rho\omega^2\widehat{u}_i+\tfrac{\partial\widehat{\sigma}_{ij}}{\partial x_j}+\widehat{f}_i \qquad \text{where} \qquad \widehat{\sigma}_{ij}(\mathbf{x},\omega)=c_{ijpq}\tfrac{\partial}{\partial x_p}\widehat{u}_q(\mathbf{x},\omega)\,.$$

Also note that $\widehat{u}_i(\mathbf{x},-\omega)=\widehat{u}_i(\mathbf{x},\omega)^*$ and $\widehat{f}_i(\mathbf{x},-\omega)=\widehat{f}_i(\mathbf{x},\omega)^*$.

(a) For any two such wavefields $\widehat{u}_i^a(\mathbf{x},\omega_a)$ and $\widehat{u}_i^b(\mathbf{x},\omega_b)$ establish that

$$\int_D\left\{-\left(\rho\omega_a^2\widehat{u}_i^a+\widehat{f}_i^a\right)\widehat{u}_i^b+\widehat{u}_i^a\left(\rho\omega_b^2\widehat{u}_i^b+\widehat{f}_i^b\right)\right\}dV=\int_{\partial\mathcal{D}}\left\{\left(\widehat{\sigma}_{ij}^a n_j\right)\widehat{u}_i^b-\widehat{u}_i^a\left(\widehat{\sigma}_{ij}^b n_j\right)\right\}dS.$$

(b) Use this expression to prove that for any single wavefield $\widehat{u}_i(\mathbf{x},\omega)$

$$\int_D\left\{\widehat{f}_i^*\widehat{v}_i+\widehat{v}_i^*\widehat{f}_i\right\}dV=\int_{\partial\mathcal{D}}\left\{-\left(\widehat{\sigma}_{ij}^* n_j\right)\widehat{v}_i-\widehat{v}_i^*\left(\widehat{\sigma}_{ij}n_j\right)\right\}dS$$

where $\widehat{v}_i(\mathbf{x},\omega)\,e^{-i\omega t}=-i\omega\widehat{u}_i(\mathbf{x},\omega)\,e^{-i\omega t}$ is the corresponding Fourier component of the velocity $v_i(\mathbf{x},t)=\frac{\partial}{\partial t}u_i(\mathbf{x},t)$. Interpret this result in terms of the time-averaged work done by the combined body forces $\widehat{f}_i(\mathbf{x},\omega)\,e^{-i\omega t}+\widehat{f}_i(\mathbf{x},-\omega)\,e^{i\omega t}$ and the corresponding outward flow of energy at the boundary $\partial\mathcal{D}$.

(c) For wavefields $\widehat{u}_i^a(\mathbf{x},\omega_a)$ and $\widehat{u}_i^b(\mathbf{x},\omega_b)$ for which $\widehat{f}_i^a(\mathbf{x},\omega_a)=0$ and $\widehat{f}_i^b(\mathbf{x},\omega_b)=0$ establish that

$$\int_D\rho\left(\widehat{u}_i^a\right)^*\widehat{u}_i^b dV=0$$

if $\omega_a^2\neq\omega_b^2$ and both wavefields satisfy traction-free boundary conditions on $\partial\mathcal{D}$. From this deduce that there are at most a countably infinite number of frequencies ω such that $\widehat{u}_i(\mathbf{x},\omega)$ can be non-zero when there are no body or boundary forces.

(d) Hence, prove that for all other frequencies the solutions to

$$0=\rho\omega^2\widehat{u}_i+\tfrac{\partial\widehat{\sigma}_{ij}}{\partial x_j}+\widehat{f}_i$$

with traction boundary conditions on $\partial\mathcal{D}$ are unique, and that for traction boundary conditions the complete wavefield $u_i(\mathbf{x},t)$ becomes unique when initial conditions $u_i(\mathbf{x},0)=0$ and $v_i(\mathbf{x},0)=0$ are applied everywhere in the domain $\mathcal{D}$.

4 In a uniform isotropic medium consider P-SV waves propagating in the (x, z) plane with x and t dependence of the form $e^{i(kx-\omega t)}$ with ω real and positive, and k also real and such that $|k| < \frac{\omega}{\alpha} \left(< \frac{\omega}{\beta}\right)$, so that both

$$k_\alpha = \sqrt{\frac{\omega^2}{\alpha^2} - k^2} \qquad \text{and} \qquad k_\beta = \sqrt{\frac{\omega^2}{\beta^2} - k^2}$$

are also real-valued.

(a) For waves involving propagation in the positive z-direction, with P and S wave potentials of the form

$$\phi_+ = A_+ e^{i(kx + k_\alpha z - \omega t)}, \qquad \psi_+ = B_+ e^{i(kx + k_\beta z - \omega t)},$$

determine the matrix Z_+ such that

$$\begin{pmatrix} \sigma_{xz} \\ \sigma_{zz} \end{pmatrix} = -Z_+ \begin{pmatrix} v_x \\ v_z \end{pmatrix},$$

and hence prove that the time-averaged energy flux in the z-direction

$$\frac{1}{4} \begin{pmatrix} v_x \\ v_z \end{pmatrix}^\dagger \left(Z_+ + Z_+^\dagger\right) \begin{pmatrix} v_x \\ v_z \end{pmatrix} \left(= \tfrac{1}{2} \mathrm{Re} \left\{-v_x^* \sigma_{xz} - v_z^* \sigma_{zz}\right\}\right)$$

is $$\frac{\rho\omega}{2(k^2 + k_\alpha k_\beta)} \left(k_\alpha |v_x|^2 + k_\beta |v_z|^2\right) (> 0),$$

where $\dagger$ denotes the transpose of the complex conjugate.

(b) Similarly, for waves involving propagation in the negative z-direction, with P and S wave potentials of the form

$$\phi_- = A_- e^{i(kx - k_\alpha z - \omega t)}, \qquad \psi_- = B_- e^{i(kx - k_\beta z - \omega t)},$$

determine the corresponding matrix Z_-, establishing that $Z_- = -Z_+^\dagger$, and prove that the time-averaged energy flux in the z-direction is

$$\frac{\rho\omega}{2(k^2 + k_\alpha k_\beta)} \left(-k_\alpha |v_x|^2 - k_\beta |v_z|^2\right) \quad (< 0).$$

(c) For a mixture of waves involving propagation in both the positive and negative z-directions, with

$$\begin{pmatrix} v_x \\ v_z \end{pmatrix} = \begin{pmatrix} v_x^+ \\ v_z^+ \end{pmatrix} + \begin{pmatrix} v_x^- \\ v_z^- \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \sigma_{xz} \\ \sigma_{zz} \end{pmatrix} = -Z_+ \begin{pmatrix} v_x^+ \\ v_z^+ \end{pmatrix} - Z_- \begin{pmatrix} v_x^- \\ v_z^- \end{pmatrix},$$

explain how v_x^+, v_z^+, v_x^-, v_z^- can be determined from v_x, v_z, σ_{xz}, σ_{zz} and obtain an explicit expression for $(Z_+ - Z_-)^{-1}$. Establish that in this case the time-averaged energy flux in the z-direction is

$$\frac{\rho\omega}{2(k^2 + k_\alpha k_\beta)} \left(k_\alpha \left[\left|v_x^+\right|^2 - \left|v_x^-\right|^2\right] + k_\beta \left[\left|v_z^+\right|^2 - \left|v_z^-\right|^2\right]\right),$$

and that this expression can be rewritten as

$$\frac{\rho\omega^3}{2} \left(k_\alpha \left[|\phi_+|^2 - |\phi_-|^2\right] + k_\beta \left[|\psi_+|^2 - |\psi_-|^2\right]\right).$$

END OF PAPER