UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 5 June, 2006 1.30 to 3.30

PAPER 31

ROUGH PATH THEORY AND APPLICATIONS

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The question carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) Let x be a Lipschitz continuous $\mathbb{R}^d$-valued path. Define $S_N(x)_{s,t}$, the *step-N signature of the path segment* $x|_{[s,t]}$. Show that the path $t \mapsto S_N(x)_{0,t}$ solves a controlled ordinary differential equation driven by x.

(ii) What is meant by *pathlevel solution to a Rough Differential Equation (RDE)?* Use Davie's lemma to prove existence of a pathlevel RDE solution.

2 Define $(G^N(\mathbb{R}^d), \otimes, {}^{-1}, e)$, the free step-$N$ nilpotent group over $\mathbb{R}^d$. State Chow's Theorem and prove it in the special case of $N = 2$ and $d = 2$. [Hint: draw a picture.]
A path with values in the step-2 nilpotent group over $\mathbb{R}^2$ is given by

$$\mathbf{y}_t = \exp\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}\right).$$

Compute $\mathbf{y}_s^{-1} \otimes \mathbf{y}_t$ and discuss the Hölder regularity of $\mathbf{y}$ with respect to the Carnot-Caratheodory metric. For what p is $\mathbf{y}$ a weak geometric p-rough path?

3 (i) Let $d \geqslant 2$ be an integer. In the context of a d-dimensional standard Brownian motion $B = (B^1, B^2, \ldots, B^d)$, define *Enhanced Brownian Motion.* Show that there is a modification of Enhanced Brownian Motion, denoted by $\mathbf{B}$, so that for any fixed $\alpha \in [0, 1/2)$,

$$\|\mathbf{B}\|_{\alpha-\text{Hölder};[0,1]} < \infty \text{ a.s.}$$

[*Integrability properties of Lévy's area and scaling properties of Enhanced Brownian Motion may be assumed.*]

(ii) Now consider the case of a 2-dimensional standard Brownian motion $B = (\beta, \tilde{\beta})$. Let $B(n) = \left(\beta(n), \tilde{\beta}(n)\right)$ be the dyadic piecewise linear approximation of level n to B, that is, $B(n)_{k/2^n}$ equals $B_{k/2^n}$ for all $k = 0, \ldots, 2^n$ and $B(n)$ is affine linear on each interval $[(k-1)/2^n, k/2^n]$, $k = 1, \ldots, 2^n$. You may assume without proof that (a) for all $t \in [0, 1]$,

$$B(n)_t = \mathbb{E}\left[B_t | B_{i/2^n}; i = 0, \ldots, 2^n\right]$$
$$\int_0^t \beta(n)\, d\tilde{\beta}(n) = \mathbb{E}\left[\int_0^t \beta d\tilde{\beta} | B_{i/2^n}; i = 0, \ldots, 2^n\right].$$

and (b) the r.v. $\|\mathbf{B}\|_{\alpha-\text{Hölder};[0,1]}$ has finite moments of all orders. Explain how martingale arguments can be used to prove that for all $t \in [0, 1]$, $\mathbf{B}(n)_t \equiv S_2(B(n))_t \to \mathbf{B}_t$ a.s. and $\sup_n \|\mathbf{B}(n)\|_{\alpha-\text{Hölder}} < \infty$ a.s..

4 (i) Define the support of a Borel probability measure on a Polish space.

(ii) State and prove the Stroock-Varadhan support theorem in uniform topology for solutions of the Stratonovich stochastic differential equation $dY = V_1(Y) \circ dB^1 + \ldots + V_d(Y) \circ dB^d$, $Y_0 = y_0 \in \mathbb{R}^e$. Here B denotes a standard Brownian motion on $\mathbb{R}^d$ and $V_1, \ldots, V_d$ are bounded vector fields on $\mathbb{R}^e$ with bounded derivatives of all orders. [You may assume that Y is given by a Rough Differential Equation driven by Enhanced Brownian Motion $\mathbf{B}$, along the vector fields $V_1, \ldots, V_d$ and started from y_0 at time 0. You may use the Universal Limit Theorem and results on Enhanced Brownian motion $\mathbf{B}$ without proof.]

END OF PAPER