UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 13 June 2001 9 to 11

PAPER 53

RENORMALISATION IN DYNAMICAL SYSTEMS

*Answer any **TWO** questions. The questions carry equal weight.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 In what follows, consider the family T_s of tent maps $T_s : [-1, 1] \to [-1, 1]$ for $1 < s \leq 2$, defined by

$$T_s : x \mapsto 1 - s|x|.$$

Also, recall that a continuous map $f : I \to I$ of a closed interval $I \subset \mathbb{R}$ has a "horseshoe" if, and only if, there is an interval $J \subset I$ and two disjoint open subintervals $K_0, K_1 \subset J$ such that $f(K_i) = J$ for $i = 0, 1$.

(a) Suppose that $\sqrt{2} < s \leq 2$. Locate the largest interval, $A \subseteq [-1, 1]$, such that $T_s(A) = A$. Express the endpoints of this interval as points on the orbit of the origin $x = 0$. What happens to the orbits of points outside A? Further, prove that any subinterval $L \subset A$ (of non-zero length) eventually expands under iteration to the whole of A, i.e. there exists some k with $T_s^k(L) = A$. Hence deduce that the set of periodic points in A is dense in A.

(b) Prove that if $\sqrt{2} < s \leq 2$, then T_s is "chaotic" in the sense that some iterate has a horseshoe (you may use a graphical argument to assist your explanation).

(c) Given that T_s is "chaotic" (in the sense used above) for $\sqrt{2} < s \leq 2$, use renormalization to show that T_s is "chaotic" for *all* $s \in (1, 2]$.

2 This question concerns Feigenbaum's explanation for the universality of period-doubling cascades observed in typical families of unimodal maps.

(a) Consider the one-parameter family $f_\mu : x \mapsto 1-\mu x^2$ on $[-1,1]$ and let $\mu = s_n$ denote the parameter value giving rise to a super-stable 2^n map, using s_∞ to denote the parameter value at the accumulation of the first period doubling cascade. Sketch the bifurcation (attracting orbit) diagram for $0 \leq \mu \leq s_\infty$. Define the universal constants α and δ.

(b) Recall that the doubling operator $\mathbf{T}$ is defined by

$$\mathbf{T}(f)(x) := a^{-1} f(f(ax)),$$

subject to the normalization condition $f(0) = 1$, for some scale factor a (with $-1 < a < 0$ and a depending on f). Explain the purpose of the normalization condition and define a so that $\mathbf{T}$ preserves this condition.

(c) Outline Feigenbaum's conjectures concerning the action of the doubling operator $\mathbf{T}$ on the space of normalized even analytic maps that are unimodal on $[-1,1]$ with a maximum of degree $d = 2$ at a critical point at the origin.

Hence give a renormalization explanation for the universality observed in the period-doubling cascades of typical one-parameter families of such maps. (You are encouraged to use diagrams in order to clarify your explanation.)

(d) Let Δ_d denote the space of normalized even analytic maps that are unimodal on $[-1,1]$ with a maximum of degree d at a critical point at the origin, where $d \geq 2$ is an even integer. Suppose that $g(x) \in \Delta_d$ is a solution to the fixed-point problem $g = \mathbf{T}(g)$. Deduce the first-order relationship

$$(a-1)da^{d-1} = 1,$$

justifying the steps in your argument. Hence, for the case $d = 2$, find a first-order approximation to the renormalization fixed point $g(x)$ and the universal constant $\alpha \approx -a^{-1}$.

3 The following question concerns the techniques used in Lanford's proof for the existence of a non-trivial fixed point g to the doubling operator $\mathbf{T}$. In what follows, $D\mathbf{\Phi}(p)$ denotes the (Fréchet) derivative (at the point p) of an operator $\mathbf{\Phi}$ acting on a suitable Banach space.

(a) Suppose that $\mathcal{X}$ is a Banach space, $\mathcal{D} \subset \mathcal{X}$ is open, that $\mathbf{\Phi} : \mathcal{X} \to \mathcal{X}$ is a C^2 map, and that $D\mathbf{\Phi}(p)$ is invertible for each $p \in \mathcal{D}$. Also, suppose that $p_0 \in \mathcal{D}$ is an *approximate zero* of $\mathbf{\Phi}$. Recall that Taylor's Theorem tells us that for p near to p_0,

$$\mathbf{\Phi}(p) \approx \mathbf{\Phi}(p_0) + [D\mathbf{\Phi}(p_0)](p - p_0).$$

Hence explain *briefly* the idea behind Newton's method for finding zeroes of $\mathbf{\Phi}$ and express the method as an operator $\mathbf{N} : \mathcal{X} \to \mathcal{X}$. Further, suppose that $\bar{p}$ is a *zero* of $\mathbf{\Phi}$. Show that $\bar{p}$ is a *fixed point* of $\mathbf{N}$ and prove that Newton's method has quadratic convergence to the fixed point $\bar{p}$ on a small enough ball around $\bar{p}$. Present an adapted method $\mathbf{A} : \mathcal{X} \to \mathcal{X}$ suitable for finding *fixed-points* (rather than zeroes) of $\mathbf{\Phi}$. Demonstrate that fixed points of $\mathbf{\Phi}$ are also fixed points of $\mathbf{A}$.

(b) Let p_0 denote an approximate *fixed point* of an adapted Newton method $\mathbf{A}$ and let $r > 0$. Suppose that $||D\mathbf{A}(p)|| \leq \kappa < 1$ for all p such that $||p - p_0|| \leq r$. Prove that the inequality,

$$||\mathbf{A}(p_0) - p_0|| \leq r(1 - \kappa),$$

ensures that there is exactly one fixed point of $\mathbf{A}$ inside the ball of functions of radius r centered on p_0. (You may assume the Contraction Mapping Theorem.)

(c) With reference to the techniques mentioned above, outline the main steps in Lanford's proof that the doubling operator $\mathbf{T}$ has a locally unique fixed point g (in a suitable space of functions) in the degree $d = 2$ case.